Exhausters — new tools in nonsmooth analysis

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Constructive Nonsmooth Analysis and Related Topics

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Dini and Hadamard derivatives

$$f(x+g) = f(x) + h(g) + o_x(g),$$

- Dini derivative

$$f'_D(x,g) = \lim_{\alpha \downarrow 0} \frac{f(x+\alpha g) - f(x)}{\alpha}.$$

$$h(g) = f'_D(x,g), \quad \lim_{\alpha \downarrow 0} \frac{o_x(\alpha g)}{\alpha} = 0 \quad \forall g \in \mathbb{R}^n.$$

- Hadamard derivative

$$f'_H(x,g) = \lim_{[\alpha, g'] \to [+0, g]} \frac{f(x+\alpha g') - f(x)}{\alpha}.$$

$$h(g) = f'_H(x,g), \quad \lim_{||g|| \to 0} \frac{o_x(g)}{||g||} = 0.$$

Dini and Hadamard derivatives. Minimum conditions

Theorem

For the point x_* to be a global or local minimizer of f on X, it is necessary that

$$f'_D(x_*, g) \geq 0 \quad \forall g \in \mathbb{S},$$

$$f'_H(x_*, g) \geq 0 \quad \forall g \in \mathbb{S}.$$

Condition

$$f'_H(x_*, g) > 0 \quad \forall g \in \mathbb{S}$$

is sufficient condition for a strict local minimum of the function f on X.

Dini and Hadamard derivatives. Maximum conditions

Theorem

For the point x^* to be a global or local maximizer of f on X, it is necessary that

$$f'_D(x^*, g) \leq 0 \quad \forall g \in \mathbb{S},$$

$$f'_H(x^*, g) \leq 0 \quad \forall g \in \mathbb{S}.$$

Condition

$$f'_H(x^*, g) < 0 \quad \forall g \in \mathbb{S}$$

is sufficient condition for a strict local maximum of the function f on X.

Quasidifferentils

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called quasidifferentiable in the sense of Dini (Hadamard) at a point $x \in \mathbb{R}^n$, if it is directionally differentiable in the sense of Dini (Hadamard) at the point x and its respective directional derivative $f'(x, g)$ is expressed in the form

$$f'(x; g) = \max_{v \in \underline{\partial} f(x)} \langle v, g \rangle + \min_{w \in \overline{\partial} f(x)} \langle w, g \rangle \qquad \forall g \in \mathbb{R}^n,$$

where $\underline{\partial} f(x) \subset \mathbb{R}^n$ and $\overline{\partial} f(x) \subset \mathbb{R}^n$ are convex compact sets. The pair if sets $\mathcal{D}f(x) = [\underline{\partial} f(x), \overline{\partial} f(x)]$ is called a quasidiffential of the function f at the point x.

Upper convex and lower concave approximations

Let a positively homogenous (p.h.) functions $h\colon \mathbb{R}^n \to \mathbb{R}$ is given. Convex p.h. function $\overline{h}\colon \mathbb{R}^n \to \mathbb{R}$ is called upper convex approximation of the function h if

$$h(g) \leq \overline{h}(g) \quad \forall g \in \mathbb{R}^n.$$

- **Pshenichny B.N.** Convex analysis and extremal problems. Moscow : Nauka, 1980. 320 p.

Concave p.h. function $\underline{h}\colon \mathbb{R}^n \to \mathbb{R}$ is called lower concave approximation of the function h, if

$$h(g) \geq \underline{h}(g) \quad \forall g \in \mathbb{R}^n.$$

Exhaustive family of approximations

The set Λ^* of upper convex approximations of the function h is called exhaustive if

$$h(g) = \inf_{\overline{h} \in \Lambda^*} \overline{h}(g) \quad \forall g \in \mathbb{R}^n. \tag{1}$$

The set Λ_* of lower concave approximations of the function h is called exhaustive if

$$h(g) = \sup_{\underline{h} \in \Lambda_*} \underline{h}(g) \quad \forall g \in \mathbb{R}^n. \tag{2}$$

- **Demyanov, V.F., Rubinov, A.M.** : "Constructive Nonsmooth Analysis", Approximation & Optimization, 7. Peter Lang, Frankfurt am Main, pp. iv+416. (1995)

Existence of exhaustive family of approximations

Theorem (Rubinov, A.M.)

Let p.h. function $h(g)\colon \mathbb{R}^n \to \mathbb{R}$ is bounded from above on B_1 i.e.

$$\sup_{g \in B_1} h(g) < +\infty,$$

where $B_1 = \{g \in \mathbb{R}^n \mid \|g\| \leq 1\}$. If h is upper semicontinuous then there exists an exhaustive family of upper convex approximations of h.
If h is lower semicontinuous and bounded from below on B_1 i.e.

$$\inf_{g \in B_1} h(g) > -\infty,$$

then there exists an exhaustive family of lower concave approximations of h.

Upper exhauster

Since every p.h. function $\overline{h}$ is convex then there exists a unique convex compact set $C(\overline{h})$ such that

$$\overline{h}(g) = \max_{v \in C(\overline{h})} \langle v, g \rangle \quad \forall g \in \mathbb{R}^n.$$

Therefore (1) can be represented as

$$h(g) = \inf_{C \in E^*} \max_{v \in C} \langle v, g \rangle \quad \forall g \in \mathbb{R}^n,$$

where $E^* = \left\{ C \subset \mathbb{R}^n | C = C(\overline{h}), \quad \overline{h} \in \Lambda^* \right\}$. The family of sets E^* is called an upper exhauster of the function h.

Lower exhauster

Analogously, since every p.h. $\underline{h}$ is concave then there exists a unique convex compact set $C(\underline{h})$

$$\underline{h}(g) = \min_{v \in \underline{C}(\underline{h})} \langle v, g \rangle \quad \forall g \in \mathbb{R}^n.$$

Therefore (2) can be represented as

$$h(g) = \sup_{C \in E_*} \min_{v \in C} \langle v, g \rangle \quad \forall g \in \mathbb{R}^n,$$

where $E_* = \{C \subset \mathbb{R}^n | C = C(\underline{h}), \quad \underline{h} \in \Lambda_*\}$. The family of sets E_* is called a lower exhauster of the function h.

The case of a Lipschitz function

It was shown that if $h(g)$ is Lipschitz then it can be written both in the form

$$h(g) = h_1(g) = \min_{C \in E^*} \max_{v \in C} \langle v, g \rangle,$$

and in the form

$$h(g) = h_2(g) = \max_{C \in E_*} \min_{v \in C} \langle v, g \rangle,$$

where the families of sets E^*, E_* are totally bounded (i.e. there exists $r > 0$, such that $C \subset B_r(0_n)$ for all $C \in E^*$ (E_*), where $B_r(0_n) \in \mathbb{R}^n$ is the ball of the radius r centered at the origin).
The pair $[E_*, E^*]$ is called a biexhauster of the function h.

- **Castellani M.** A Dual Representation for Proper Positively Homogeneous Functions. // Journal of Global Optimization, Vol. 16, No. 4, 2000. P. 393–400.

Definition of exhausters

Denote $h(g) = f'(x, g)$, where $f'(x, g)$ is a directional derivative. We will work with the representation of the form

$$h(g) = \min_{C \in E^*} \max_{v \in C} \langle v, g \rangle,$$

$$h(g) = \max_{C \in E_*} \min_{v \in C} \langle v, g \rangle,$$

where E^*, E_* are families of convex compact sets from $\mathbb{R}^n$ which are called upper and lower exhauster respectively.

- **Demyanov V.F.** Exhausters of a positively homogeneous function. Optimization 45(1–4), 13–29 (1999)
- **Demyanov V.F.**Exhausters and convexificators – new tools in nonsmooth analysis. // Quasidifferentiability and related topics, P. 85–137, Nonconvex Optim. Appl., 43, Kluwer Acad. Publ., Dordrecht, (2000)

Calculus of exhausters

Let $E = [E^*, E_*]$ take $\lambda \in \mathbb{R}$ and define

$$\lambda E = \begin{cases} [\lambda E^*, \lambda E_*], \lambda \geq 0, \\ [\lambda E_*, \lambda E^*], \lambda < 0. \end{cases}$$

If $E_1 = [E_1^*, E_{1*}]$, $E_2 = [E_2^*, E_{2*}]$ assume

$$E_1 + E_2 = [E_1^* + E_2^*, E_{1*} + E_{2*}].$$

Theorem

Consider functions f_1 and f_2. Let pairs $E(f_1(x_0)) = [E^*(f_1(x_0)), E_*(f_1(x_0))]$ and $E(f_2(x_0)) = [E^*(f_2(x_0)), E_*(f_2(x_0))]$ be exhausters at a point x_0 of the functions f_1 and f_2 respectively. Then $E(f(x_0)) = \lambda_1 E(f_1(x_0)) + \lambda_2 E(f_2(x_0))$ is an exhauster of the function $f = \lambda_1 f_1 + \lambda_2 f_2$ at a point x_0, where $\lambda_1, \lambda_2 \in \mathbb{R}$.

Theorem

Consider functions f_1 *and* f_2*. Let pairs* $E_1(f_1(x_0)) = [E_1^*(f_1(x_0)), E_{1*}(f_1(x_0))]$ *and* $E_2(f_2(x_0)) = [E_2^*(f_2(x_0)), E_{2*}(f_2(x_0))]$ *be exhausters at a point* x_0 *of the functions* f_1 *and* f_2 *respectively. Assume a function* $F = f_1 f_2$*. Then*

$$E(F(x_0)) = f_1(x_0)E_2(f_2(x_0)) + f_2(x_0)E_1(f_1(x_0)),$$

Theorem

Consider functions f_1 *and* f_2*. Let pairs* $E_1(f_1(x_0)) = [E_1^*(f_1(x_0)), E_{1*}(f_1(x_0))]$ *and* $E_2(f_2(x_0)) = [E_2^*(f_2(x_0)), E_{2*}(f_2(x_0))]$ *be exhausters at a point* x_0 *of the functions* f_1 *and* f_2 *respectively. Assume function* $F = f_1/f_2$ *(if* $f_2(x_0) \neq 0$ *in the latter case). Then*

$$E(F(x_0)) = \frac{1}{f_2^2(x_0)} \left(f_2(x_0)E_1(f_1(x_0)) - f_1(x_0)E_2(f_2(x_0))\right)$$

Theorem

Let functions $\phi_1(x), \ldots, \phi_m(x) \colon \mathbb{R}^n \to \mathbb{R}$ *be directionally differentiable at a point* x_0, *a function* $F(y) = F(y_1, \ldots, y_m) \colon \mathbb{R}^m \to \mathbb{R}$ *be continuously differentiable at the point* $y_0 = (\phi_1(x_0), \ldots, \phi_m(x_0))$ *and let* E_i *be biexhausters of the functions* $h_i(g) = \phi_i'(x_0, g)$. *Then the function* $f(x) = F(\phi_1(x), \ldots, \phi_m(x))$ *is also directionally differentiable at* x_0 *and there exists a biexhauster of the function* $h(g) = f'(x_0, g)$ *of the form*

$$E(H) = \sum_{i=1}^{m} \frac{\partial F(y_0)}{\partial y_i} E_i.$$

Theorem

Consider functions f_i, $i \in I = 1:N$ *and let pairs* $E(f_i(x_0)) = [E^*(f_i(x_0)), E_*(f_i(x_0))]$ *be exhauster at a point* x_0 *of the functions* f_i, $i \in I$. *Assume a function* $F_1 = \max_{i \in I} f_i$, $F_2 = \min_{i \in I} f_i$. *Then the lower exhauster of the function* F_1 *at the point* x_0 *has the form*

$$E_*(F_1(x_0)) = \bigcup_{i \in R(x_0)} E_*(f_i(x_0)),$$

while the upper exhauster of the function F_2 *at the point* x_0 *has the form*

$$E^*(F_2(x_0)) = \bigcup_{i \in Q(x_0)} E^*(f_i(x_0)),$$

where $R(x) = \{i \in I \mid f_i(x) = F_1(x)\}$, $Q(x) = \{i \in I \mid f_i(x) = F_2(x)\}$.

Quasidifferentiable functions

Theorem

Let a function $f(x)$ is quasidifferentiable at point x_0 and $[\underline{\partial} f(x), \overline{\partial} f(x)]$ is the quasidifferential of the function f at the point x_0. Then there exists an upper and a lower exhauster of the form

$$E^* = \left\{ C = w + \underline{\partial} f(x_0) \,\middle|\, w \in \overline{\partial} f(x_0) \right\},$$

$$E_* = \left\{ C = v + \overline{\partial} f(x_0) \,\middle|\, v \in \underline{\partial} f(x_0) \right\},$$

$$h(g) = f_1'(x_0, g) = \min_{w \in \overline{\partial} f_1(x_0)} \langle w, g \rangle + \max_{v \in \underline{\partial} f_1(x_0)} \langle v, g \rangle =$$

$$= \min_{w \in \overline{\partial} f_1(x_0)} \left[\langle w, g \rangle + \max_{v \in \underline{\partial} f_1(x_0)} \langle v, g \rangle \right] = \min_{w \in \overline{\partial} f_1(x_0)} \max_{v \in \underline{\partial} f_1(x_0)} \langle w + v, g \rangle =$$

$$= \min_{w \in \overline{\partial} f_1(x_0)} \max_{v \in w + \underline{\partial} f_1(x_0)} \langle v, g \rangle = \min_{C \in E_1^*} \max_{v \in C} \langle v, g \rangle.$$

Conditions of minimum in terms of proper exhausters

Theorem

For the inequality

$$h(g) := \min_{C \in E^*} \max_{v \in C} \langle v, g \rangle \geq 0 \quad \forall g \in \mathbb{S},$$

$$\left(h(g) := \min_{C \in E^*} \max_{v \in C} \langle v, g \rangle > 0 \quad \forall g \in \mathbb{S} \right),$$

where E^* is the family of convex compact sets from $\mathbb{R}^n$, to hold it is necessary and sufficient that

$$0_n \in C \quad \forall C \in E^*.$$

$$\left(0_n \in \text{int}\, C \quad \forall C \in E^*\right).$$

Directions of steepest descent in terms of proper exhausters and quasidifferentials

Suppose that the necessary minimum conditions are not satisfied. In this case there is $C \in E^*$ such that $0_n \notin C$. Define

$$d(C) := \min_{v \in C} ||v|| = ||v_C|| > 0, \quad g_C = -\frac{v_C}{||v_C||}.$$

Let for the set $\widehat{C}$ we have

$$d(\widehat{C}) = \max_{C \in E^*} d(C).$$

Then $\widehat{g} = g_{\widehat{C}}$ is a direction of the steepest descent.
Let $w_0 \in \overline{\partial} h$, $v_0 \in \underline{\partial} h$ are such that

$$\max_{w \in \overline{\partial} h} \min_{v \in \underline{\partial} h} \|v + w\| = \min_{v \in \underline{\partial} h} \|v + w_0\| = \|v_0 + w_0\|,$$

Then $\widehat{g} = -(v_0 + w_0)\|v_0 + w_0\|^{-1}$ is a direction of steepest descent.

Conditions of maximum in terms of proper exhausters

Theorem

For the inequality

$$h(g) := \max_{C \in E_*} \min_{v \in C} \langle v, g \rangle \leq 0 \quad \forall g \in \mathbb{S},$$

$$\left(h(g) := \max_{C \in E_*} \min_{v \in C} \langle v, g \rangle < 0 \quad \forall g \in \mathbb{S} \right),$$

where E_* is the family of convex compact sets from $\mathbb{R}^n$, to hold it is necessary and sufficient that

$$0_n \in C \quad \forall C \in E_*.$$

$$(0_n \in \operatorname{int} C \quad \forall C \in E_*).$$

Directions of steepest ascent in terms of proper exhausters and quasidifferentials

Suppose that the necessary maximum conditions are not satisfied.
In this case there is $C \in E_*$ such that $0_n \notin C$. Denote

$$d(C) := \min_{v \in C} ||v|| = ||v_C|| > 0, \quad g_C = \frac{v_C}{||v_C||}.$$

Let for the set $\widehat{C}$ we have

$$d(\widehat{C}) = \max_{C \in E^*} d(C).$$

Then $\widehat{g} = g_{\widehat{C}}$ is a direction of the steepest ascent.
Let $w_1 \in \overline{\partial} h$, $v_1 \in \underline{\partial} h$ are such that

$$\max_{v \in \underline{\partial} h} \min_{w \in \overline{\partial} h} \|v + w\| = \min_{w \in \overline{\partial} h} \|w + v_1\| = \|v_1 + w_1\|,$$

Then $\widehat{g} = (v_1 + w_1)\|v_1 + w_1\|^{-1}$ is a direction of steepest ascent.

Convertors

Let we have an upper or a lower exhauster E of p.h. function h. Define

$$C'(g) = \ cl \ co \left\{ v(C) \in C \mid \langle v(C), g \rangle = \max_{v \in C} \langle v, g \rangle, \quad C \in E \right\}.$$

Denote

$$E^{\diamond} = \left\{ C = C'(g) \mid g \in \mathbb{S} \right\}.$$

Then

- $(E_*)^{\diamond}$ is an upper exhauster of h,
- $(E^*)^{\diamond}$ is a lower exhauster of h.

Convertors. Example

Consider a function $h\colon \mathbb{R}^2 \to \mathbb{R}$,

$$h(g) = \min_{i=1:2} \max_{v \in C_i} \langle v, g \rangle,$$

where $C_1 = \operatorname{co}\{(1,1),(-1,1)\}$, $C_2 = \{(0,-1)\}$.
The family of sets $E = \{C_1, C_2\}$ is an upper exhauster of h.

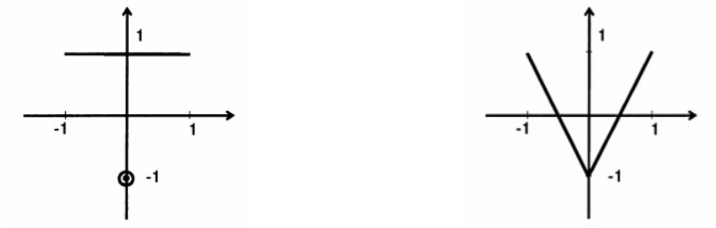

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Figure: (a) upper exhauster E^*; (b) converted upper exhauster $(E^*)^\diamond$ for the function h.

Convertors. Example

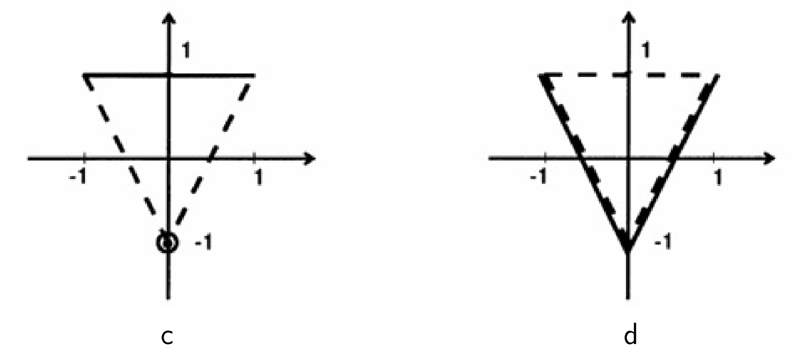

Figure: (c) $(E^*)^{\diamond^2}$; (d) $(E^*)^{\diamond^3}$.

Conditions of minimum in terms of adjoint exhausters

Theorem

For the inequality

$$h(g) := \max_{C \in E_*} \min_{v \in C} \langle v, g \rangle \geq 0 \quad \forall g \in \mathbb{S},$$

$$\left(h(g) := \max_{C \in E_*} \min_{v \in C} \langle v, g \rangle > 0 \quad \forall g \in \mathbb{S} \right),$$

where E_* is the family of convex compact sets from $\mathbb{R}^n$, to hold it is necessary and sufficient that for all $g \in \mathbb{S}$ there exists $C(g) \in E_*$ such that for every $v \in C(g)$ the inequality $\langle v, g \rangle \geq 0$ $(\langle v, g \rangle > 0)$ is valid.

Directions of steepest descent in terms of adjoint exhausters

The fulfillment of the necessary condition for minimum means that for any hyperplane passing through the origin there is a set from E_* such that this hyperplane separates the given set from the origin. If this is not the case, then there is $g \in \mathbb{S}$ such that

$$h(g) := \max_{C \in E_*} \min_{v \in C} \langle v, g \rangle = -a(g) < 0.$$

Direction g is a direction of descent. Denote the set of all such directions as G and find

$$\min_{g \in G} \{-a(g)\} = -a(g_0) < 0.$$

Direction g_0 is a direction of the steepest descent while the value $-a(g_0)$ is the rate of the steepest descent.

Conditions of maximum in terms of adjoint exhausters

Theorem

For the inequality

$$h(g) := \min_{C \in E^*} \max_{v \in C} \langle v, g \rangle \le 0 \quad \forall g \in \mathbb{S},$$

$$\left(h(g) := \min_{C \in E^*} \max_{v \in C} \langle v, g \rangle < 0 \quad \forall g \in \mathbb{S} \right),$$

where E^* is the family of convex compact sets from $\mathbb{R}^n$, to hold it is necessary and sufficient that for all $g \in \mathbb{S}$ there exists $C(g) \in E^*$ such that for every $v \in C(g)$ the inequality $\langle v, g \rangle \ge 0$ $(\langle v, g \rangle > 0)$ is valid.

Directions of steepest ascent in terms of adjoint exhausters

The fulfillment of the necessary condition for minimum means that for any hyperplane passing through the origin there is a set from E^* such that this hyperplane separates the given set from the origin. If this is not the case, then there is $g \in \mathbb{S}$ такое, что

$$h(g) := \min_{C \in E^*} \max_{v \in C} \langle v, g \rangle = a(g) > 0.$$

Direction g is a direction of ascent. Denote the set of all such directions as G and find

$$\max_{g \in G} \{a(g)\} = a(g_1) > 0.$$

Direction g_1 is a direction of the steepest ascent while the value $a(g_1)$ is the rate of the steepest ascent.

Example 1

Let us consider a function $f_1 : \mathbb{R}^2 \to \mathbb{R}: \; f_1(x) = f(x_1, x_2) = |\,|x_1| - |x_2|\,|$ at a point $x_0 = 0_2 = (0, 0)$. We have $f(x_0) = 0$ and

$$
\begin{aligned}
f(x_0 + g) &= f(x_0) + \max\{|g_1| - |g_2|; |g_2| - |g_1|\} = \\
&= \max\Big\{ \max\{g_1, -g_1\} - \max\{g_2, -g_2\}; \max\{g_2, -g_2\} - \max\{g_1, -g_1\} \Big\} = \\
&= \max\Big\{ \max\{g_1, -g_1\} + \max\{g_1, -g_1\}; \max\{g_2, -g_2\} + \max\{g_2, -g_2\} \Big\} - \\
&\qquad - \max\Big\{g_1, -g_1\Big\} - \max\Big\{g_2, -g_2\Big\} = \\
&= \max\Big\{ \max\{2g_1, -2g_1\}; \max\{2g_2, -2g_2\} \Big\} + \min\Big\{g_1, -g_1\Big\} + \\
&\qquad + \min\Big\{g_2, -g_2\Big\} = \\
&= \max\Big\{2g_1, -2g_1, 2g_2, -2g_2\}\Big\} + \\
&\qquad + \min\Big\{g_1 + g_2; g_1 - g_2; -g_1 + g_2; -g_1 - g_2\Big\}.
\end{aligned}
$$

Therefore the expansion

$$\begin{aligned} f_1(x_0+g) = f_1(x_0) + \min \Big\{ & \max\{-g_1+g_2; 3g_1+g_2; g_1-g_2; g_1+3g_2\}; \\ & \max\{-3g_1+g_2; g_1+g_2; -g_1-g_2; -g_1+3g_2\}; \\ & \max\{-3g_1-g_2; g_1-g_2; -g_1-3g_2; -g_1+g_2\}; \\ & \max\{-g_1-g_2; 3g_1-g_2; g_1-3g_2; g_1+g_2\} \Big\} = \\ & = f_1(x_0) + \min_{C\in E^*(x_0)} \max_{v\in C} (v, g) \end{aligned}$$

is valid, for all $g \in \mathbb{R}^2$, where the upper exhauster has the form

$$E^*(x_0) = \begin{cases} C_1 = \mathrm{co}\{(1,3),(3,1),(1,-1),(-1,1)\}, \\ C_2 = \mathrm{co}\{(-1,3),(1,1),(-1,-1),(-3,1)\}, \\ C_3 = \mathrm{co}\{(-1,1),(1,-1),(-1,-3),(-3,-1)\}, \\ C_4 = \mathrm{co}\{(1,1),(3,-1),(1,-3),(-1,-1)\}. \end{cases}$$

Similarly we get

$$f_1(x_0+g) = f_1(x_0) + \max\Big\{ \min\{3g_1+g_2; 3g_1-g_2; g_1+g_2; g_1-g_2\},$$
$$\min\{g_1+3g_2; g_1+g_2; -g_1+3g_2; -g_1+g_2\}$$
$$\min\{-g_1+g_2; -g_1-g_2; -3g_1+g_2; -3g_1-g_2\},$$
$$\min\{g_1-g_2; g_1-3g_2; -g_1-g_2; -g_1-3g_2\}\Big\} =$$
$$= f_1(x_0) + \max_{C\in E_*(x_0)} \min_{v\in C}(v,g)$$

is valid, for all $g\in\mathbb{R}^2$, where

$$E_*(x_0) = \begin{cases} C_5 = \mathrm{co}\{(3,1),(3,-1),(1,-1),(1,1)\}, \\ C_6 = \mathrm{co}\{(1,3),(-1,3),(-1,1),(1,1)\}, \\ C_7 = \mathrm{co}\{(-3,1),(-3,-1),(-1,1),(-1,-1)\}, \\ C_8 = \mathrm{co}\{(1,-1),(-1,-3),(1,-3),(-1,-1)\}. \end{cases}$$

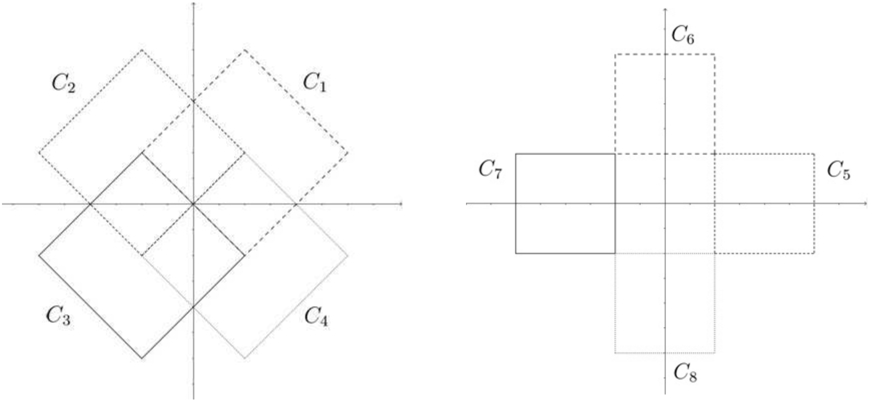

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Figure: Upper $E^*(x_0)$ (a) and lower $E_*(x_0)$ (b) exhauster for the function f_1.

Example 2

Let us consider a function $f_2 : \mathbb{R}^2 \to \mathbb{R}$: $f_1(x) = f(x_1, x_2) = |x_1| + |x_2|$ at a point $x_0 = 0_2 = (0, 0)$. The representation

$$f_2(x_0 + g) = f_2(x_0) + \min_{C \in E^*(x_0)} \max_{v \in C}(v, g) \ \forall g \in \mathbb{R}^2,$$

is valid, where the upper exhauster has the form

$$E^*(x_0) = co\{(-1, -1), (-1, 1), (1, -1), (1, 1)\}.$$

Also we have

$$f_2(x_0 + g) = f_2(x_0) + \max_{C \in E_*(x_0)} \min_{v \in C}(v, g)$$

where the lower exhauster has the form

$$E_*(x_0) = \Big\{(-1, -1); (-1, 1); (1, -1); (1, 1)\Big\}.$$

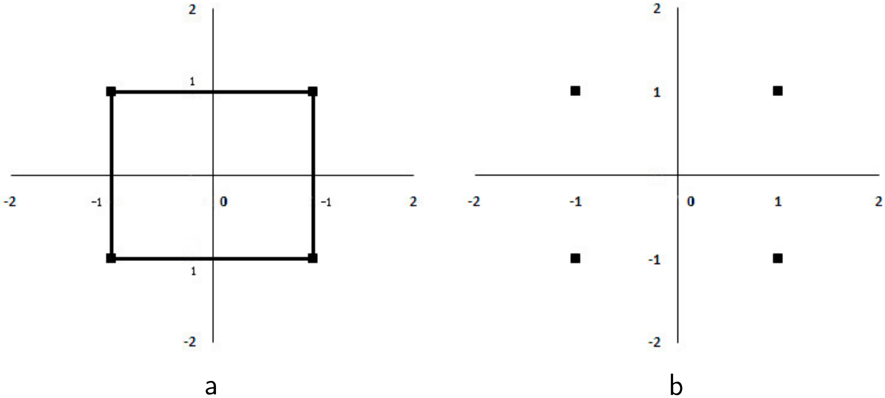


Figure: Upper $E^*(x_0)$ (a) and lower $E_*(x_0)$ (b) exhauster for the function f_2.

Example 3

$$h(g_1, g_2) = \min\Big\{ \max\{g_1 + g_2, g_1 - g_2, -g_1 + g_2, -g_1 - g_2\}\,; \\ 3g_1 + 2g_2; g_1 - 2g_2; -3g_1 + g_2; -2g_1 - g_2\Big\}.$$

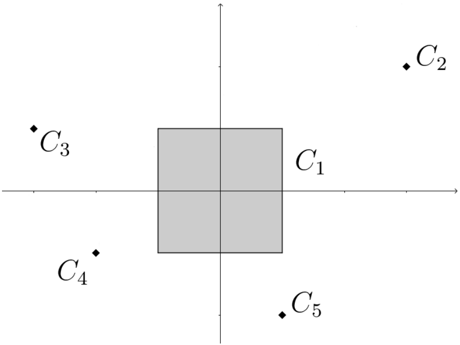


Figure: An upper exhauster E^* of the function h.

Example 3

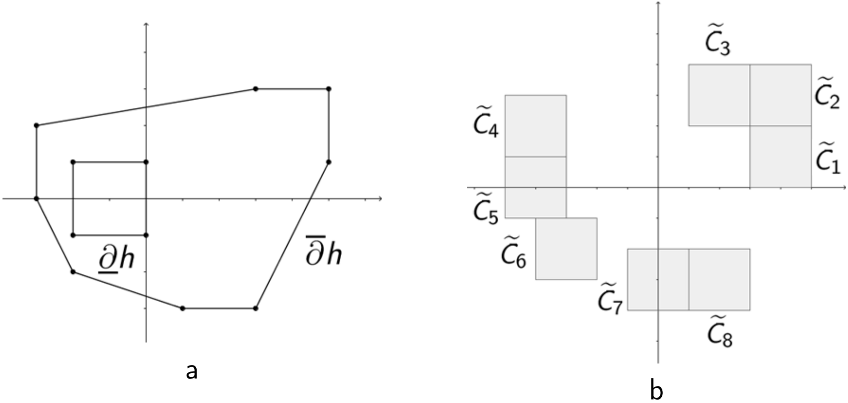


Figure: Quasidifferential (a) and built via it the upper exhauster (b) of the function h.

Minimality of exhausters

Minimality by inclusion

The problem of minimality of exhausters was first studied by Vera Roshchina. The easiest way to determine the minimality is based on the quantity of sets that exhausters contain.

Definition

We say that the upper (lower) exhauster $E_1(h)$ of the p.h. function h is smaller by inclusion than the other upper (lower) exhauster $E_2(h)$ of the same function h, if $E_1(h) \subset E_2(h)$.

Definition

We say that the upper (lower) exhauster $E(h)$ of the p.h. function h is the minimal by inclusion if there is no other upper (lower) exhauster $\widetilde{E}(h)$ of the function h so that $\widetilde{E}(h) \subset E(h)$.

Minimality of exhausters

Minimality by inclusion

However, it is clear that such a definition does not affect the structure of sets themselves, although the structure plays a major role in many cases. Indeed consider the function $h_1 : \mathbb{R} \to \mathbb{R}$, $h_1(x) = |x|$. It is obvious that

$$E_1^*(h_1) = \{[-2, 1], [-1, 2]\}$$

is an upper exhauster of our function. None of the sets included in $E_1^*(h_1)$ can be discarded, because in that case the family will no longer be an exhauster of the h_1. Nevertheless, each of sets from $E_1^*(h_1)$ can be reduced to the interval $[-1, 1]$, which is the subdifferential of h_1. Since the subdifferential of a convex function is uniquely defined, then there is only one upper exhauster $E_2^*(h_1) = \{[-1, 1]\}$, consisting of the single set. This exhauster is "smaller" than $E_1^*(h_1)$.
This example demonstrates the need for a different definition of the minimal exhauster, the definition which would reflect not only the minimality by inclusion, but also the minimality by shape of sets included in an exhauster.

Minimality of exhausters

Minimality by shape

Definition

We say that the upper (lower) exhauster $E_1(h)$ of the p.h. function h is smaller by shape than the other upper (lower) exhauster $E_2(h)$ of the same function h, if

$$\forall \widetilde{C} \in E_1(h) \; \exists C \in E_2(h) \colon \; \widetilde{C} \subset C.$$

Definition

We say that the upper (lower) exhauster $E(h)$ of the p.h. function h is the minimal by shape if there is no other upper (lower) exhauster $\widetilde{E}(h)$ of the function h so that

$$\forall \widetilde{C} \in \widetilde{E}(h) \; \exists C \in E(h) \colon \; \widetilde{C} \subset C.$$

Note that an exhauster, minimal by shape, is also minimal by inclusion, but the converse is not true.

Geometric conditions of minimality

Notation used

For an arbitrary $g \in \mathbb{R}^n$ and $C \in E$ define a supporting hyperplane

$$L_{C,g}(x) = \langle x - v_C, g \rangle = 0, \quad v_C = \underset{v \in C}{\operatorname{argmax}} \langle v, g \rangle,$$

such that C lays in the closed negative half-space

$$H_{-}(C, g) = \{x \in \mathbb{R}^n \mid L_{C,g}(x) \leq 0\},$$

generated by this hyperplane ($C \subset H_{-}(C, g)$).

Geometric conditions of minimality

Minimality by inclusion

Theorem

Let E be an (upper or lower) exhauster of a p.h. function $h\colon \mathbb{R}^n \to \mathbb{R}$ and $\widetilde{C} \in E$. For $\widetilde{E} = E \setminus \{\widetilde{C}\}$ to be also (upper or lower respectively) exhausters of the function h it is necessary and sufficient that for any $g \in \mathbb{S}$ there is a $C_g \in \widetilde{E}$, such that $C_g \subset H_-(\widetilde{C}, g)$.

This theorem means that for the set $\widetilde{C}$ to be discarded it is necessary and sufficient that in closed negative half-space of any supporting hyperplane of the set $\widetilde{C}$, passing through an arbitrary extreme point of $\widetilde{C}$, lays at least one another set from E.

Geometric conditions of minimality

Minimality by inclusion

As a corollary from the theorem we get necessary and sufficient condition, that a set belonging to an exhausters cannot be discarded.

Theorem

Let E be an (upper or lower) exhauster of a p.h. function $h\colon \mathbb{R}^n \to \mathbb{R}$. For the impossibility of $\widetilde{C} \in E$ to be discarded from an (upper or lower respectively) exhauster (i.e. that the family $\widetilde{C} \in E = E \setminus \{\widetilde{C}\}$ is not (upper or lower respectively) exhauster of function h) it is necessary and sufficient that there is $\widetilde{g} \in \mathbb{S}$ for which

$$\forall C \in \widetilde{E} \quad \exists v(C) \in C\colon v(C) \notin H_{-}(\widetilde{C}, \widetilde{g}).$$

Note that proposed conditions can be used for cleaning exhausters from "unnecessary" sets.

Minimality by inclusion

Example 1

Consider the function $h\colon \mathbb{R}^2 \to \mathbb{R}$,

$$h(g_1, g_2) = \min \left\{ \max \{g_1 + g_2, g_1 - g_2, -g_1 + g_2, -g_1 - g_2\} ; \right. \\ \left. 3g_1 + 2g_2; g_1 - 2g_2; -3g_1 + g_2; -2g_1 - g_2 \right\}.$$

It is obvious that an upper exhauster of this function has the form $E^* = \{C_1, C_2, C_3, C_4, C_5\}$, where

$$C_1 = \mathrm{co}\,\{(1,1),(1,-1),(-1,1),(-1,-1)\}\,,\;\; C_2 = \{(3,2)\}\,, \\ C_3 = \{(-3,1)\}\,,\;\; C_4 = \{(-2,-1)\}\,,\;\; C_5 = \{(1,-2)\}\,.$$

Minimality by inclusion

Example 1

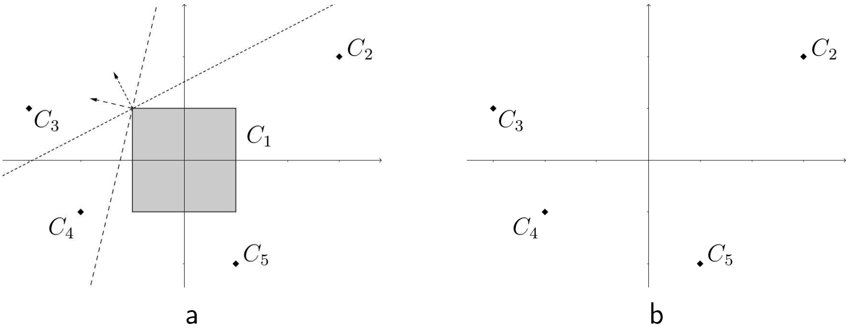


Figure: An upper exhauster E^* (a) and reduced upper exhauster $\widetilde{E}^*$ (b) of the function h from example 1.

$$h(g_1, g_2) = \min \{3g_1 + 2g_2; g_1 - 2g_2; -3g_1 + g_2; -2g_1 - g_2\}.$$

Minimality by inclusion

Example 2

Consider the function $h\colon \mathbb{R}^2 \to \mathbb{R}$,

$$h(g_1, g_2) = \min\left\{\max_{v\in\mathbb{S}} \langle v, g\rangle\,; 2g_1 + 2g_2; 2g_1 - 2g_2; -2g_1 + 2g_2; -2g_1 - 2g_2\right\}$$

Its upper exhauster has the form $E^* = \{C_1, C_2, C_3, C_4, C_5\}$, where

$$C_1 = \operatorname{co}\left\{(g_1, g_2) \in \mathbb{R}^2 \middle|\ g_1^2 + g_2^2 \le 1\right\},\ \ C_2 = \{(2, 2)\}\,,$$
$$C_3 = \{(-2, 2)\}\,,\ \ C_4 = \{(-2, -2)\}\,,\ \ C_5 = \{(2, -2)\}\,.$$

Minimality by inclusion

Example 2

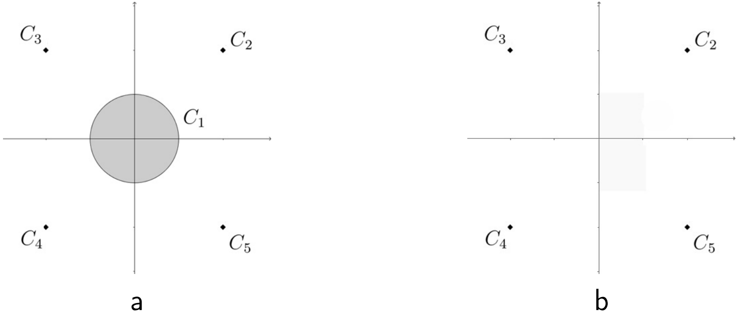


Figure: An upper exhauster E^* (a) and reduced upper exhauster $\widetilde{E}^*$ (b) of the function h from example 2.

$$h(g_1, g_2) = \min\{2g_1 + 2g_2; 2g_1 - 2g_2; -2g_1 + 2g_2; -2g_1 - 2g_2\}.$$

Proceed to the question of minimality by shape.
Let $E = \{C_\omega \mid \omega \in \Omega\}$ and there is a subset Θ of the index set Ω, such that $\bigcap\limits_{\omega\in\Theta} C_\omega = \widehat{C} \neq \emptyset$. Denote $B = \{C_\omega \mid \omega \in \Theta\}$.

Theorem

Let E be an (upper or lower) exhauster of a p.h. function $h\colon \mathbb{R}^n \to \mathbb{R}$, for all $g \in \mathbb{S}$ there is a $\omega_g \in \Theta$, such that $C_{\omega_g} \subset H_-(\widehat{C}, g)$. Then a family $\widetilde{E} = \{E \setminus B\} \bigcup \widehat{C}$ is also (upper or lower respectively) exhauster of the function h.

This theorem allows to obtain necessary condition of impossibility of replacing sets C_ω, $\omega \in \Omega$ in an exhauster, with their intersection.

Theorem

For the impossibility to replace sets C_ω, $\omega \in \Omega$ in an exhauster of a p.h. function h, with their intersection (i.e. for the family $\{E \setminus B\} \bigcup \widehat{C}$ not to be an exhauster of the h) it is necessary that

$$\exists \widetilde{g} \in \mathbb{S} \colon \forall \omega \in \Theta \quad C_\omega \not\subset H_-(\widehat{C}, \widetilde{g}).$$

In other words, there must be a supporting hyperplane of $\widehat{C}$, for which there is no set C_ω, $\omega \in \Omega$ that fully lays in the closed negative half-space of this hyperplane.

Minimality by shape. Work with the intersection of sets

Example 3

Consider the function $h\colon \mathbb{R}^2 \to \mathbb{R}$,

$$\begin{aligned} h(g_1, g_2) = \min \Big\{ & \max\{-g_1 + g_2, -g_1 - g_2, 3g_1 + g_2, 3g_1 - g_2\}; \\ & \max\{g_1 + g_2, g_1 - g_2, -3g_1 + g_2, -3g_1 - g_2\}; \\ & -2g_1 + 2g_2 \Big\}, \end{aligned}$$

which upper exhausters has the form $E^* = \{C_1, C_2, C_3\}$, where

$$\begin{aligned} C_1 &= \text{co}\{(-1,1), (-1,-1), (3,1), (3,-1)\}, \\ C_2 &= \text{co}\{(1,1), (1,-1), (-3,1), (-3,-1)\}, \\ C_3 &= \{(-2,2)\}. \end{aligned}$$

Minimality by shape. Work with the intersection of sets

Example 3

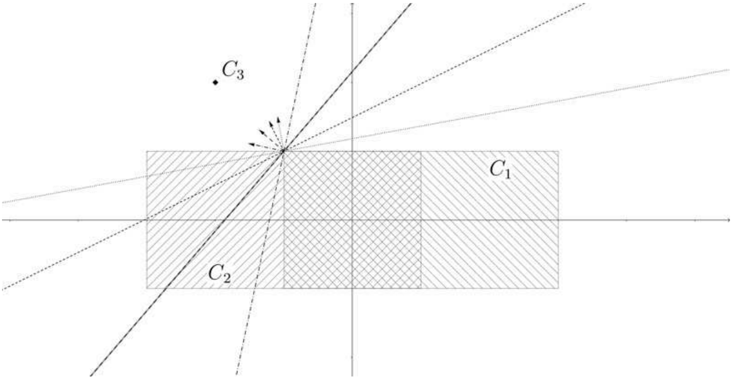


Figure: An upper exhauster E^* of the function h from example 3.

Minimality by shape. Work with the intersection of sets

Example 3

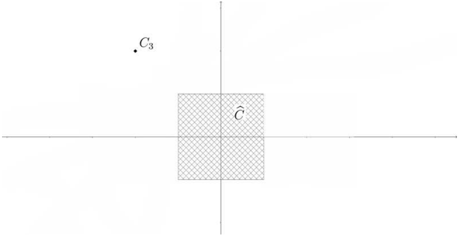


Figure: Reduced upper exhauster $\widetilde{E}^*$ of the function h from example 3.

Thus, the function

$$h(g_1, g_2) = \min\{\max\{-g_1 + g_2, -g_1 - g_2, 3g_1 + g_2, 3g_1 - g_2\};\ \max\{g_1 + g_2, g_1 - g_2, -3g_1 + g_2, -3g_1 - g_2\};\ -2g_1 + 2g_2\}.$$

can be represented in the form

$$h(g_1, g_2) = \min\{\max\{-g_1 + g_2, -g_1 - g_2, g_1 + g_2, g_1 - g_2\};\ -2g_1 + 2g_2\}.$$

Minimality by shape. Work with the intersection of sets
Example 3 (V.A. Roschina)

Note that all examples discussed in works of Roschina can be solved by means of proposed geometric conditions of reduction.
Let p.h. function h has an upper exhauster E^*, consisting of all unit balls tangential to the origin. It is obvious that conditions of the previous theorem is satisfied, and after reduction procedure we get a family $\overline{E}^* = \{\{0\}\}$, which is evidently minimal exhauster.

Minimality by shape. Discarding vertices

Now proceed to the important practical case. Let a family E consists of convex polytopes.
Choose an arbitrary $C \in E$. Denote vertices of a set C by $v_i(C)$, where $i \in I_C$, I_C is the set of indices. It is clear that $C = \operatorname{co}\{v_i(C) \mid i \in I_C\}$.
Define a cone

$$\Gamma_{v_i(C)} = \left\{ g \in \mathbb{R}^n \;\middle|\; \max_{v \in C} \langle v, g \rangle = \langle v_i(C), g \rangle \right\}$$

of directions, which are normal to the supporting hyperplanes of the set C at the point $v_i(C)$. Obviously $\bigcup_{i \in I_C} \Gamma_{v_i(C)} = \mathbb{R}^n$.

Minimality by shape

Necessary condition for discarding vertices

Theorem

Let E be an (upper or lower) exhauster of a p.h. function $h\colon \mathbb{R}^n \to \mathbb{R}$, $\widetilde{C} = \operatorname{co}\left\{ v_i(\widetilde{C}) \;\middle|\; i \in I_{\widetilde{C}} \right\} \in E$, $\widehat{C} = \operatorname{co}\left\{ v_i(\widetilde{C}) \;\middle|\; i \in I_{\widetilde{C}} \setminus \{\widehat{i}\} \right\}$. For the family $\widetilde{E} = \left\{ E \setminus \{\widetilde{C}\} \right\} \bigcup \widehat{C}$ to be also (upper or lower respectively) exhauster of the function h, it is necessary that

$$\forall g \in \Gamma_{v_{\widehat{i}}(\widetilde{C})} \quad \exists C_g \in E, C_g \neq \widetilde{C}\colon\ C_g \subset H_-(\widetilde{C}, g). \tag{3}$$

In other words, for every supporting hyperplane of the set $\widetilde{C}$, passing through the vertex $v_{\widehat{i}}(\widetilde{C})$, there must be at least one set from E besides $\widetilde{C}$, that fully lays in the closed negative half-space of this hyperplane.

Minimality by shape

Sufficient condition for discarding vertices

Theorem

Let E be an (upper or lower) exhauster of a p.h. function $h\colon \mathbb{R}^n \to \mathbb{R}$, $\widetilde{C} = \operatorname{co}\left\{ v_i(\widetilde{C}) \;\middle|\; i \in I_{\widetilde{C}} \right\} \in E$, $\widehat{C} = \operatorname{co}\left\{ v_i(\widetilde{C}) \;\middle|\; i \in I_{\widetilde{C}} \setminus \{\widehat{i}\} \right\}$. For a family $\widetilde{E} = \left\{ E \setminus \{\widetilde{C}\} \right\} \bigcup \widehat{C}$ to be also (upper or lower respectively) exhauster of the function h, it is sufficient that

$$\forall g \in \Gamma_{v_{\widehat{i}}(\widetilde{C})}\ \exists C_g \in E, C_g \neq \widetilde{C}, \exists u \in \mathbb{R}^n \colon \begin{cases} \langle v - u, g \rangle \geq 0 & v \in \widetilde{C} \\ \langle v - u, g \rangle \leq 0 & v \in C_g \end{cases} \tag{4}$$

This theorem means that to discard the vertex $v_{\widehat{i}}$ of the set $\widetilde{C}$, it is sufficient that for any supporting hyperplane of $\widetilde{C}$, passing through $v_{\widehat{i}}$, there is some parallel hyperplane, that separates $\widetilde{C}$ and some other set from E in the sense (4).

Minimality by shape. Discarding vertices
Example 5

Consider the function $h\colon \mathbb{R}^2 \to \mathbb{R}$,

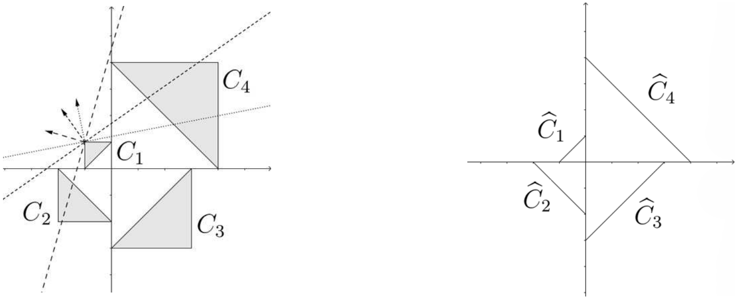
$$h(g_1, g_2) = \min\Big\{ \max\{-g_1, g_2, -g_1 + g_2\}\,;\max\{-2g_1, -2g_2, -2g_1 - 2g_2\}\,; \\ \max\{3g_1, -3g_2, 3g_1 - 3g_2\}\,;\max\{4g_1, 4g_2, 4g_1 + 4g_2\}\Big\}.$$

Its upper exhauster has the form $E^* = \{C_1, C_2, C_3, C_4\}$, where

$$C_1 = \operatorname{co}\{(-1,0),(0,1),(-1,1)\}\,,\ \ C_2 = \operatorname{co}\{(-2,0),(0,-2),(-2,-2)\}\,, \\ C_3 = \operatorname{co}\{(3,0),(0,-3),(3,-3)\}\,,\ \ C_4 = \operatorname{co}\{(4,0),(0,4),(4,4)\}\,.$$

Minimality by shape. Discarding vertices

Example 5



a) b)

Figure: An upper exhauster E^* (a) and reduced upper exhauster $\widetilde{E}^*$ (b) of the function h from example 5.

Minimality by form. Discarding vertices

Example 5

Thus, the function

$$h(g_1, g_2) = \min \{ \max \{-g_1, g_2, -g_1 + g_2\} ; \max \{-2g_1, -2g_2, -2g_1 - 2g_2\} ; \\ \max \{3g_1, -3g_2, 3g_1 - 3g_2\} ; \max \{4g_1, 4g_2, 4g_1 + 4g_2\} \}.$$

can be rewritten in a more compact form

$$h(g_1, g_2) = \min \{ \max \{-g_1, g_2\} ; \max \{-2g_1, -2g_2\} ; \\ \max \{3g_1, -3g_2\} ; \max \{4g_1, 4g_2\} \}.$$