Scientific Research of Prof. V.F. Demyanov

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Constructive Nonsmooth Analysis and Related Topics, May 22–27, 2017, Saint-Petersburg

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- The results named after prof. V.F. Demyanov are pointed out.
- We stress the same details and results as prof. V.F. Demyanov did in his lectures.

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Note that both functions $f'_{\mathcal{D}}(x, \cdot)$ and $f'_{\mathcal{H}}(x, \cdot)$ are **positively homogeneous**!

Notation

Prof. Demyanov always pointed out that, in the general case, one has

$$\lim_{[g',\alpha]\to[g,+0]} \frac{f(x+\alpha g') - f(x)}{\alpha} \neq \lim_{g'\to g,\alpha\to+0} \frac{f(x+\alpha g') - f(x)}{\alpha}$$

Therefore we must use the notation $[g',\alpha] \to [g,+0]$.[1]

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Note that if f is Lipschitz continuous near x, then

$$f'_{\mathcal{D}}(x, g) = f'_{\mathcal{H}}(x, g) \quad \forall g \in \mathbb{R}^n.$$

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Thus, first order optimality conditions for the cases of maximum and minimum do not coincide. This is also impossible in the smooth case.

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If x is not a local minimizer, then $f'(x, g^*) \leqslant 0$. Suppose that $f'(x, g^*) < 0$. Then the function f descreases in the direction g^*, i.e. $f(x + \alpha g^*) \leqslant f(x)$ for any sufficiently small $\alpha > 0$.

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Similar definitions can be given for the case of steepest ascent.

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In other words, we would like to develop a **constructive** theory of directional derivatives (**constructive nonsmooth analysis**).

Figure: Constructive Nonsmooth Analysis!

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Danskin-Demyanov Theorem

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Is f directionally differentiable? How to compute its directional derivative?

Theorem 1 (Danskin-Demyanov, 1966)

Let f_i, $1 \leqslant i \leqslant m$, be differentiable at x. Then the function

$$f(\cdot) = \max_{1 \leqslant i \leqslant m} f_i(\cdot)$$

is directionally differentiable at the point x and

$$f'(x, g) = \max_{i \in R(x)} \langle \nabla f_i(x), g \rangle \quad \forall g \in \mathbb{R}^n,$$

where $R(x) = \{ i \in 1 \colon m \mid f_i(x) = f(x) \}$.

Subdifferential of the Max-Function

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$$f'(x, g) > 0 \quad \forall g \in \mathbb{R}^n \setminus \{0\} \iff 0 \in \operatorname{int} \partial f(x).$$

Furthermore, the direction of steepest descent $g^* = -v^*/\|v^*\|$, where

$$v^* \in \operatorname*{arg\,min}_{v \in \partial f(x)} \|v\|.$$

MDM-method

Thus, in order to find the direction of steepest descent one has to solve the problem

$$\min \|v\|^2 \quad \text{subject to} \quad v \in C,$$

where $C = \mathrm{co}\{a_1, \ldots, a_s\} \subset \mathbb{R}^n$ is the convex hull of a finite set of points $\{a_1, \ldots, a_s\}$.

[2] Mitchell, B.F., Demyanov, V.F., Malozemov, V.N. (1971). Finding the point of a polyhedron closest to the origin. Vestnik Leningr. State Univ., vol. 19, pp. 38–45. (In Russian)

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In 1971, the well-known Mitchell-Demyanov-Malozemov (MDM) method for solving this problem was proposed.[2,3]

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Figure: Prof. V.N. Malozemov and Prof. V.F. Demyanov on the occasion of 40th Anniversary of the MDM method.

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Pshenichnyi Subdifferential[4]

Definition 3 (Pshenichnyi, 1969)

A function f defined in a neighbourhood of a point x is called subdifferentiable in the sense of Pschenichnyi (P-subdifferentiable) at this point if f is directionally differentiable at x, and there exists a nonempty compact convex set $\partial f(x)$ such that

$$f'(x,g) = \max_{v \in \partial f(x)} \langle v, g \rangle \quad \forall g \in \mathbb{R}^n.$$

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Clearly, all max-functions and all convex functions are P-subdifferentiable. On the other hand, for any P-subdifferentiable function f, the function $h(x) = -f(x)$ is not P-subdifferentiable.

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Quasidifferential

In 1979, Prof. V.F. Demyanov (jointly with A.M. Rubinov and L.N. Polyakova) introduced a class of quasidifferentiable functions.[5]

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Definition 4

A function f defined in a neighbourhood of a point x is called is called **quasidifferentiable** at this point, if f is directionally differentiable at x, and there exists nonempty compact convex sets $\underline{\partial} f(x)$ and $\overline{\partial} f(x)$ such that

$$f'(x, g) = \max_{v \in \underline{\partial} f(x)} \langle v, g \rangle + \min_{w \in \overline{\partial} f(x)} \langle w, g \rangle \quad \forall g \in \mathbb{R}^n.$$

The pair $\mathscr{D} f(x) = [\underline{\partial} f(x), \overline{\partial} f(x)]$ is called a **quasidifferential** of f at x. The set $\underline{\partial} f(x)$ is referred to as the **subdifferential** of f at x, while the set $\overline{\partial} f(x)$ is referred to as the **superdifferential** of f at x.

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The Smallest Quasidifferential

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Clearly, a quasidifferential is not unique. If $\mathscr{D}f(x) = [\underline{\partial}f(x), \overline{\partial}f(x)]$ is a quasidifferential of f at x, then the pair $[\underline{\partial}f(x) + C, \overline{\partial}f(x) - C]$, where C is a compact convex set, is a quasidifferential of f at x as well.

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Open Problem 1

How to find the smallest (in some sence) quasidifferential $\mathscr{D}f(x)$?

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and

$$\alpha[U_1, V_1] = \begin{cases} [\alpha U_1, \alpha V_1], & \text{if } \alpha \geqslant 0, \\ [\alpha V_1, \alpha U_1], & \text{if } \alpha < 0. \end{cases}$$

(the same operations as in the Minkowski-Rådstöm-Hörmander space).

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A similar formula holds true for $f(x) = \min_{1 \leqslant i \leqslant m} f_i(x)$.

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$$f'(x, g) = \max_{v \in \underline{\partial} f(x)} \langle v, g \rangle + \min_{w \in \overline{\partial} f(x)} \langle w, g \rangle = \min_{w \in \overline{\partial} f(x)} \max_{v \in \underline{\partial} f(x) + \{w\}} \langle v, g \rangle.$$

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Thus, $f'(x,g) \geqslant 0$ for all $g \in \mathbb{R}^n$ iff

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Furthermore, directions of steepest descent have the form $-v^*/\|v^*\|$, where v^* is a solution of the problem

$$\max_{w\in\overline{\partial}f(x)}\min_{v\in\underline{\partial}f(x)+\{w\}}\|v\|^2.$$

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Definition 5

The Demyanov difference $U \dot{-} V$ of the sets U and V is defined as follows

$$U \dot{-} V = \operatorname{cl}\operatorname{co}\left\{ \nabla p_U(x) - \nabla p_V(x) \;\middle|\; x \in \Omega \right\}.$$

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Let f be locally Lipschitz continuous and quasidifferentiable at a point x. Under some additional assumptions of the function f the following inclusions hold true

$$\underline{\partial} f(x) \dot{-} (-\overline{\partial} f(x)) \subseteq \partial_{Cl} f(x) \subseteq \underline{\partial} f(x) + \overline{\partial} f(x).$$

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Thus, one can estimate the Clarke subdifferential with the use of a quasidifferential.[8]

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All subdifferential mappings are discontinuous in the nonsmooth case as well!

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for some $R > 0$ in order to find a line search direction v^*. Note that in this case one uses the **nonhomogeneous** approximation of the max-function, which is the maximum of **affine** functions.

Definition 6 (Demyanov, 1988)

Let a function f be defined in a neighbourhood of x. The function f is called **codifferentiable** at x if there exist compact convex sets $\underline{d}f(x)$, $\overline{d}f(x) \subset \mathbb{R}^{n+1}$ such that

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Without loss of generality one can suppose that

$$\max_{(a,v) \in \underline{d}f(x)} a = \min_{(b,w) \in \overline{d}f(x)} b = 0.$$

Figure: Codifferentials!

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The continuity of a codifferential mapping is preserved under all standard operations!

A Connection with Quasidifferentials

If f is codifferentiable at x, then f is quasidifferentiable at x, and

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Theorem 2 (Demyanov, 2000)

Let $h(g) = f'(x, g)$ be continuous. Then a minimal convexificator of $h(g)$ is unique iff $h(g)$ is either convex or concave (in this case $C^* = \underline{\partial} h(0)$ or $C^* = \overline{\partial} h(0)$).

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Another Approach: Upper Convex Approximations[14]

Definition 9 (Pschenichnyi, 1980)

A sublinear function $p(g)$ is called an **upper convex approximation**(u.c.a.) of the function $h(g) = f'(x, g)$, if $h(g) \leqslant p(g)$ for all $g \in \mathbb{R}^n$.
Similary, a superlinear function $q(g)$ is called a **lower concave approximation**(l.c.a.) of $h(g)$, if $h(g) \geqslant q(g)$ for all $g \in \mathbb{R}^n$.

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Exhaustive Families of U.C.A. and L.C.A.[15,16]

Definition 10 (Rubinov, 1981)

A family $\{p_\lambda\}$, $\lambda \in \Lambda$, of u.c.a. of $h(g)$ is called **exhaustive**, if

$$h(g) = \inf_{\lambda \in \Lambda} p_\lambda(g) \quad \forall g \in \mathbb{R}^n.$$

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Exhaustive families of u.c.a. and l.c.a. provide a simple and complete description of $f'(x, g)$. But when do such families exist?

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Theorem 3 (Rubinov, 1995)

Let the function $h(g) = f'(x, g)$ be continuous. Then there exist an exhaustive family of u.c.a. of $h(g)$ and an exhaustive famliy of l.s.c. of $h(g)$.

[17] Demyanov, V.F., Rubinov, A.M. (1995) Constructive Nonsmooth Analysis. Peter Lang, Frankfurt am Main.

Exhausters[18]

Suppose that f is quasidifferentiable at x. Then one has

$$f'(x, g) = \min_{w \in \overline{\partial} f(x)} p_w(g), \quad p_w(g) = \max_{v \in \underline{\partial} f(x) + \{w\}} \langle v, g \rangle$$

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Definition 11 (Demyanov, 1999)

A family $E_*(h)$ of convex compact set is called an **upper exhauster** of the function $h(g) = f'(x,g)$, if

$$h(g) = \inf_{C \in E_*(h)} \max_{v \in C} \langle v, g \rangle \quad \forall g \in \mathbb{R}^n.$$

Similarly, a family $E^*(h)$ of convex compact sets is called a **lower exhauster** of $h(g)$, if

$$h(g) = \sup_{C \in E^*(h)} \min_{w \in C} \langle w, g \rangle \quad \forall g \in \mathbb{R}^n.$$

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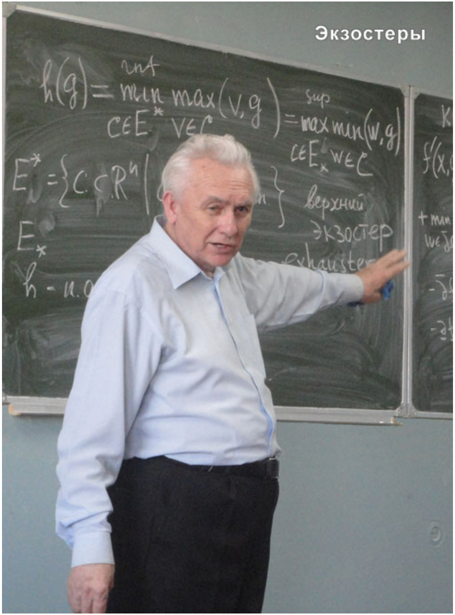

Figure: Exhausters!

Existence and Optimality Conditions

Clearly, if $h(g) = f'(x, g)$ is continuous, then there exist both upper and lower exhausters of $h(g)$.

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Thus, exhausters solve the original problem, since they provide constructive optimality conditions and exist in most particular cases!

Converters and Demyanov-Ryabova Conjecture[19]

Let E be a bounded family of compact convex sets.

[19] Danilidis, A., Petitjean, C. (2017) A partial answer to the Demyanov-Ryabova conjecture. arXiv: 1702.00168.

Converters and Demyanov-Ryabova Conjecture[19]

Let E be a bounded family of compact convex sets. For any $g \in S^1$ (unit sphere) define

$$C(g) = \operatorname{cl}\operatorname{co}\left\{ w(C) \in C \;\middle|\; \langle w(C), g \rangle = \min_{w \in C} \langle w, g \rangle, \quad C \in E \right\}.$$

Denote $E^{\diamond} = \{ C(g) \mid g \in S^1 \}$. The operator $\diamond$ is called the **convertor**.

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Denote $E^{\diamond} = \{C(g) \mid g \in S^1\}$. The operator $\diamond$ is called the **convertor**. If E is an upper (lower) exhauster of $h(g)$, then $E^{\diamond}$ is a lower (upper) exhauster of $h(g)$, while $E^{\diamond\diamond}$ is again an upper (lower) exhauster of $h(g)$ that is, in general, different from E. What will happen if we apply converator infinitely many times? Does the convertor have a fixed point?

[19] Danilidis, A., Petitjean, C. (2017) A partial answer to the Demyanov-Ryabova conjecture. arXiv: 1702.00168.

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Conjecture (Demyanov–Ryabova, 2000)

If the convertor is applied to a family of polyhedral sets sufficiently many times, then the process will stabilise with a 2-cycle.

[19] Danilidis, A., Petitjean, C. (2017) A partial answer to the Demyanov-Ryabova conjecture. arXiv: 1702.00168.

Table of Contents

From Codifferential to Coexhauster[20,21]

Let f be codifferentiable at x. Then

$$f(x+\Delta x) - f(x) = \min_{(b,w)\in \overline{d} f(x)} \max_{(a,v)\in \underline{d} f(x) + \{(b,w)\}} \big(a + \langle v, \Delta x\rangle\big) + o(\Delta x).$$

[20] Aban'kin A.E. (1998) Unconstrained minimization of H-hyperdifferentiable functions. Comput. Math. Math. Phys., vol. 38, pp. 1439–1446.

[21] Demyanov, V.F. (1999) Exhausters of a positively homogeneous function. Optim., vol. 45, pp. 13–29.

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By analogy with the definition of exhausters, one can define **coexhausters**.

Definition 12 (Aban'kin, 1998; Demyanov, 1999)

A family $\overline{E}(f(x))$ of convex compact subsets of $\mathbb{R}^{n+1}$ is called a (generalized) **lower coexhauster** of f at x, if

$$f(x+\Delta x)-f(x)=\inf_{C\in\overline{E}(f(x))}\ \max_{(a,v)\in C}(a+\langle v,\Delta x\rangle)+o(\Delta x).$$

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[20] Aban'kin A.E. (1998) Unconstrained minimization of H-hyperdifferentiable functions. Comput. Math. Math. Phys., vol. 38, pp. 1439–1446.

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Figure: Coexhausters!

Existence Theorem

Theorem 4

Let f be u.s.c. in a neighbourhood of x. Then there exists a generalized lower coexhauster of f at x.

[22] Rubinov, A.M. (2000) Abstract Convexity and Global Optimization. Kluwer Academic Publishers, Dordrecht.

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Let f be u.s.c. in a neighbourhood of x. Then there exists a generalized lower coexhauster of f at x.

This result is a simple corollary to some fundamental theorems from abstract convexity theory.[22]

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Let us note that optimality conditions can be written in terms of coexhausters, and one can construct a minimization method utilizing coexhausters.

[22] Rubinov, A.M. (2000) Abstract Convexity and Global Optimization. Kluwer Academic Publishers, Dordrecht.

Figure: Understand?