Nonsmooth Speed-Gradient Algorithms for Control of Oscillatory Systems

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Table of Contents

Table of Contents

Problem Formulation

Consider the controlled system

$$\dot{x} = f(x, u, t), \quad t \geq 0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of the system state, and $u \in \mathbb{R}^m$ is the control.

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We pose the control problem as finding a control law

$$u = U\{x(s), u(s) \colon 0 \leq s \leq t\}$$

which ensures the control objective

$$Q(x(t), t) \to 0 \quad \text{as} \quad t \to +\infty, \tag{2}$$

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where $Q(x, t)$ is a nonnegative smooth **goal function**.[1]

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Speed-Gradient Algorithm in Differential Form

To design a control algorithm, compute the scalar function

$$\omega(x, u, t) = \frac{d}{dt} Q(x, t) = \frac{\partial Q(x, t)}{\partial t} + \left(\frac{\partial Q(x, t)}{\partial x} \right)^T f(x, u, t).$$

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The function ω is the **speed of change** of $Q(x, t)$ along trajectories of the system.

Calculate the gradient of ω with respect to u

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$$\nabla_u \omega(x, u, t) = \left(\frac{\partial f(x, u, t)}{\partial u} \right)^T \frac{\partial Q(x, t)}{\partial x}, \qquad (3)$$

and take the control algorithm in the form

$$\frac{du}{dt} = -\Gamma \nabla_u \omega(x, u, t), \qquad (4)$$

where $\Gamma = \Gamma^T > 0$.

Speed-Gradient Algorithm in Finite Form

One can consider the algorithm in the finite form

$$u = u_0 - \Gamma \nabla_u \omega(x(t), u, t), \qquad (5)$$

where u_0 is an initial value of the control variable.

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One can consider the algorithm in the finite form

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where u_0 is an initial value of the control variable.

We also introduce the more general **Speed-Pseudogradient (SPG) algorithm**

$$u = u_0 - \gamma \psi(x(t), u, t), \tag{6}$$

where $\gamma > 0$ is a scalar gain, and the vector function ψ satisfies **the acute angle condition**

$$\psi(x, u, t)^T \nabla_u \omega(x, u, t) \geq 0. \tag{7}$$

Properties of Smooth Speed-Gradient Algorithms

Assumptions

1. *there exists a solution* $u = \kappa(x, u_0, t)$ *of the equation* $u = u_0 - \gamma\psi(x, u, t)$*, and* κ *is locally bounded uniformly in* t*;*

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4. *there exist an "ideal control law"* $u_* \colon \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^m$ *and a positive definite* $\rho \colon \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\omega(x, u_*(x, t), t)) \leq -\rho(Q(x, t)) \quad \forall t \geq 0;$$

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$$\omega(x, u_*(x, t), t)) \leq -\rho(Q(x, t)) \quad \forall t \geq 0;$$
5. $\psi(x, u, t)^T \nabla_u \omega(x, u, t) \geq \beta |\nabla_u \omega(x, u, t)|$ *for some* $\beta > 0$.

Properties of Smooth Speed-Gradient Algorithms[2,3]

Theorem 1 (Convergence of the SPG Algorithm)

Let assumptions 1–5 be valid. Then for any initial conditions $(x(0), u_0)$ *there exists* $\gamma^* \geq 0$ *such that for any* $\gamma > \gamma^*$ *a solution* $(x(t), u(t))$ *of the system* (1), (6) *is defined and bounded on* $\mathbb{R}_+$ *and the control goal*

$$\lim_{t \to +\infty} Q(x(t), t) = 0. \tag{8}$$

is achieved.

[2] Fradkov A.L. (1979) Speed-gradient scheme and its application in adaptive control problems. Autom. Remote Control, vol. 40, pp. 1333–1342.

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Table of Contents

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Let the goal function $Q(x, t)$ be locally Lipschitz continuous and directionally differentiable.

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Let the goal function $Q(x,t)$ be locally Lipschitz continuous and directionally differentiable. Choose a convex in u function $\omega(x,u,t)$ such that

$$\frac{d}{dt}Q(x,t) = Q'(x,t;f(x,u,t),1) \le \omega(x,u,t), \tag{9}$$

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where $Q'(x,t; f(x,u,t),1)$ is the directional derivative of $Q(x,t)$ at the point (x,t) in the direction $(f(x,u,t),1)$.

Define the control algorithm as follows

$$\frac{du}{dt} \in -\Gamma \partial_u \omega(x,u,t), \tag{10}$$

where $\Gamma = \Gamma^T > 0$ and $\partial_u \omega(x,u,t)$ is the subdifferential of the function $u \to \omega(x,u,t)$.

Speed-Pseudosubgradient Algorithm

We also consider the algorithm in the finite form

$$u \in u_0 - \Gamma \partial_u \omega(x, u, t), \tag{11}$$

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and the more general **Speed-Pseudosubgradient (SPSG) algorithm** (or nonsmooth SPG)

$$u = u_0 - \gamma \psi(x, u, t), \tag{12}$$

where the function ψ satisfies **the extended acute angle condition**: for any $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $t \geq 0$ there exists $v \in \partial_u \omega(x, u, t)$ such that

$$\psi(x, u, t)^T \cdot v \geq 0 \tag{13}$$

Properties of Nonsmooth Speed-Gradient Algorithms

Assumptions

1. *there exists a solution* $u = \kappa(x, u_0, t, \gamma)$ *of the equation* $u = u_0 - \gamma\psi(x, u, t)$, *and* κ *is locally bounded unifromly in* t*;*

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$$\omega(x, u_*(x, t), t)) \leq -\rho(Q(x, t)) \quad \forall t \geq 0.$$

4. *there exists* $\beta > 0$ *such that for some* $v \in \partial_u \omega(x, u, t)$ *one has*

$$\psi(x, u, t)^T \cdot v \geq \beta |v|. \tag{14}$$

Properties of Nonsmooth Speed-Gradient Algorithms[4]

Theorem 2 (Convergence of the Nonsmooth SPG Algorithm)

Let assumptions 1–4 be valid. Then there exists $\gamma^* \geq 0$ *such that for any* $\gamma > \gamma^*$ *a solution* $(x(t), u(t))$ *of the system* (1), (12) *is defined and bounded on* $\mathbb{R}_+$ *and the control goal*

$$\lim_{t \to +\infty} Q(x(t), t) = 0.$$

is achieved.

[4] Dolgopolik, M.V., Fradkov, A.L. (2017). Nonsmooth and discontinuous speed-gradient algorithms. Nonlinear Anal.: Hybrid Syst., vol. 25, pp. 99–113.

Table of Contents

Pendulum with Moving Suspension Point

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Consider a pendulum, whose suspension point can be moved along the horizontal axis. Let the control variable be the acceleration along this axis.

The motion of the pendulum is described by the equations

$$\dot{q} = \frac{1}{ml^2} \cdot p, \qquad \dot{p} = -mgl \cdot \sin q + ml \cdot u \cdot \cos q, \tag{15}$$

where q, p are generalized coordinate and momentum, u is the control action, m, l and g are the mass of the pendulum, the length of the pendulum and the gravity acceleration, respectively.

Control Goal

Let the control goal be to make the upright equilibrium globally attracitve.

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Let the control goal be to make the upright equilibrium globally attracitve. Introduce the nonsmooth objective function $Q(q,p)$ as follows

$$Q(q,p) = |H(q,p) - H_*|, \tag{16}$$

where

$$H(q,p) = \frac{1}{2ml^2} \cdot p^2 + mgl \cdot (1 - \cos q) \tag{17}$$

is the total energy of the unforced pendulum, and $H_* = H(\pi, 0) = 2mgl$ is the total energy at the upright equilibrium.

Nonsmooth SG Algorithm

If $H(q,p) \neq H_*$, then

$$Q'\Big((q,p); F(q,p,u)\Big) \quad = \quad \operatorname{sign}(H(q,p) \; - \; H_*) \; \cdot \; \frac{u}{l} \; \cdot \; p \; \cdot \; \cos q. \quad (18)$$

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Note that $u(q,p) = \{0\}$ whenever $p = 0$ or $q = \frac{\pi}{2} + 2\pi k$, $k \in \mathbb{Z}$. Therefore only the inequality $\omega(x,u,t) \leq 0$ holds true.

Behaviour of the Closed-Loop System

With the use of the LaSalle's invariance principle, one can prove that the following result holds true.[5]

[5] Dolgopolik, M.V., Fradkov, A.L. (2017). Nonsmooth and discontinuous speed-gradient algorithms. Nonlinear Anal.: Hybrid Syst., vol. 25, pp. 99–113.

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Theorem 3

For any $\gamma > 0$ *and any initial conditions* $(q(0), p(0)) \neq (0, 0)$ *all solutions of the closed-loop system are defined and bounded on* $\mathbb{R}_+$, *and have a unique* ω*-limit point* $(\pi, 0)$. *Thus, the upright equilibrium* $(\pi, 0)$ *is a unique almost global attractor of the closed-loop system.*

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Remark 1

One can verify that for any $\varepsilon > 0$ there exists $\gamma_* > 0$ such that for any $\gamma \in (0, \gamma_*)$ one has $|u(p, q)| < \varepsilon$.

[5] Dolgopolik, M.V., Fradkov, A.L. (2017). Nonsmooth and discontinuous speed-gradient algorithms. Nonlinear Anal.: Hybrid Syst., vol. 25, pp. 99–113.

Table of Contents

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$$u = -\Gamma g(x, t), \tag{20}$$

while the nonsmooth SPG algorithm takes the form

$$u = -\gamma \psi(x, u, t), \tag{21}$$

where $g(x, t)^T \psi(x, u, t) \geq 0$.

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Let $C \subset \mathbb{R}^n$ be a set of "bad" initial conditions.

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2. *for any* $x(0) \in \mathbb{R}^n \setminus C$ *there exists an abolutely continuous solution* $x(t)$ *of* (1), (21) *that is defined on* $\mathbb{R}_+$ *and* $x(t) \notin C$ *for all* $t \geq 0$;

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3. $Q(x, t)$ *is radially unbounded and continuous uniformly in* t;
4. *for any* $\Delta > 0$ *and* $r > 0$ *there exists* $a > 0$ *such that* $|g(x, t)| \geq a$ *for all* $x \in \mathbb{R}^n \setminus c$ *and* $t \geq 0$ *such that* $Q(x, t) \geq \Delta$ *and* $|x| \leq r$.

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5. *there exists a positive definite* $\rho \colon \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$g(x, t)^T \psi(x, u, t) \geq \rho(|g(x, t)|);$$

Properties of Nonsmooth Speed-Gradient Algorithms[6]

Theorem 4 (Convergence of the Nonsmooth SPG Algorithm)

Let assumptions 1–5 be valid. Then for any $x(0) \in \mathbb{R}^n \setminus C$ *and* $\gamma > 0$ *the solution* $(x(t), u(x, t, \gamma))$ *of the system* (1), (21) *is bounded on* $\mathbb{R}_+$ *and the control goal*

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is achieved.

[6] Dolgopolik, M.V., Fradkov, A.L. (2016). Speed-Gradient Control of the Brockett Integrator. SIAM J. Control Optim., vol. 54, pp. 2116–2131.

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is achieved.

Remark 2

Suppose that $|\psi(x, u, t)| \leq c < +\infty$ for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $t \geq 0$. Then choosing $\gamma > 0$ sufficiently small one can guarantee that the inequality $|u| = \gamma|\psi(x, u, t)| < \varepsilon$ holds true for an arbitrarily small prespecified $\varepsilon > 0$.

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Table of Contents

Brockett Integrator

The Brockett integrator of nonholonomic integrator

$$\begin{cases} \dot{x}_1 = u_1, \\ \dot{x}_2 = u_2, \\ \dot{x}_3 = x_1 u_2 - x_2 u_1. \end{cases} \tag{22}$$

[7] Clarke, F. (2010). Discontinuous feedback and nonlinear systems, in Proceedings of the IFAC Conference on Nonlinear Control (NOLCOS), Bologna, pp. 1–29.

Brockett Integrator

The Brockett integrator of nonholonomic integrator

$$\begin{cases} \dot{x}_1 = u_1, \\ \dot{x}_2 = u_2, \\ \dot{x}_3 = x_1 u_2 - x_2 u_1. \end{cases} \tag{22}$$

Let $\varepsilon > 0$ be arbitrarily small. Let us also impose the contraint on control

$$u_1^2 + u_2^2 \leq \varepsilon,$$

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Note that $Q(x)$ is a control Lyapunov function for (22).[7]

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Brockett Integrator

Applying the nonsmooth SPG Algorithm one gets

$$u(x) = \begin{cases} 0 & \text{if } x = 0, \\ -\gamma \frac{1}{\sigma(x)} (x_1, x_2)^T & \text{if } x_3 = 0, \sigma(x) \neq 0, \\ \gamma v(x_3) & \text{if } \sigma(x) = 0, x_3 \neq 0, \\ -\gamma \frac{1}{|g(x)|} g(x) & \text{if } \sigma(x) \neq 0, x_3 \neq 0, \end{cases} \tag{24}$$

where $|v(x_3)| = 1$, $g(x) = (g_1(x), g_2(x))$ and

$$g_1(x) = 2x_1 - 4x_2x_3 - 2\frac{|x_3|x_1}{\sigma(x)} + 2\,\text{sign}(x_3)x_2\sigma(x) \tag{25}$$

$$g_2(x) = 2x_2 + 4x_1x_3 - 2\frac{|x_3|x_2}{\sigma(x)} - 2\,\text{sign}(x_3)x_1\sigma(x). \tag{26}$$

Note that $|u(x)| \leq \gamma$ (hence $|u(x)|^2 \leq \varepsilon$, if $\gamma \leq \sqrt{\varepsilon}$).

Brockett Integrator

The proposed control law stabilizes the Brockett integrator, and it is **continuous** along solutions of the closed-loop system, provided $\sigma(x(0)) \neq 0$![8]

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$$\dot{x}_3 = -\frac{2\gamma\sigma^2(x)}{|g(x)|}(2x_3 - \sigma(x)), \quad \frac{d}{dt}\sigma(x) = -\frac{2\gamma}{|g(x)|}(\sigma(x) - x_3). \tag{27}$$

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$$\frac{d}{dt}\sigma(x) < 0 \quad \text{if} \quad \sigma(x(t)) > x_3(t),$$

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while

$$\frac{d}{dt}\sigma(x) > 0 \quad \text{if} \quad \sigma(x(t)) < x_3(t).$$

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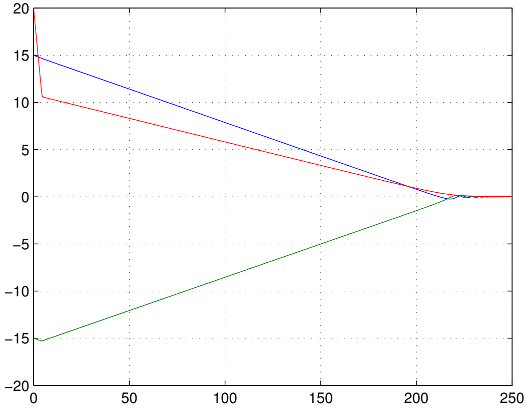

Figure: Simulation with $\gamma = 0.1$ and $x(0) = (15; -15; 20)$.

Brockett Integrator

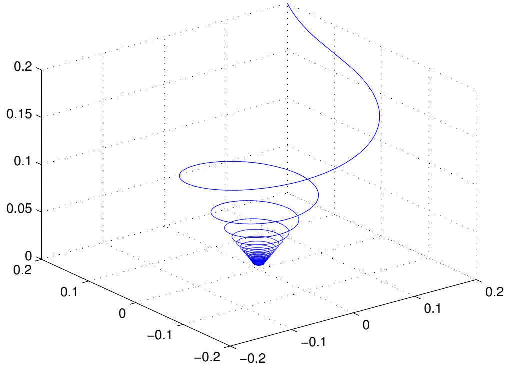

Figure: The trajectory of the closed-loop system with $\gamma = 0.1$ and $x(0) = (0,2; 0,2; 0,2)$

Brockett Integrator

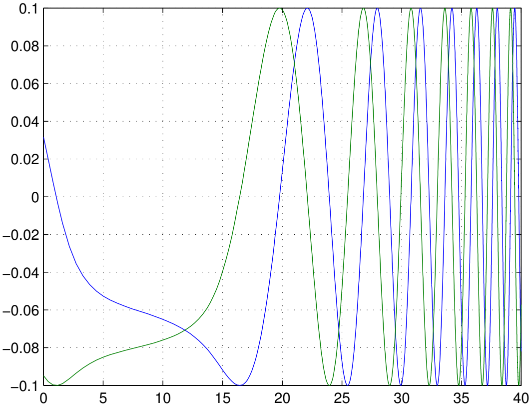

Figure: The control with $\gamma = 0.1$ and $x(0) = (0,2;0,2;0,2)$

The End!