A Stable Solution of Linear Programming Problems with the Approximate Matrix of Coefficients

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International conference
«CONSTRUCTIVE NONSMOOTH ANALYSIS AND RELATED TOPICS»
dedicated to the memory of professor V.F. Demyanov
St. Petersburg,
May 22-27, 2017

(Prehistory): Tikhonov's Regularized method of least squares – the normal solution of the approximate SLAE [1]

The regularized method of least squares (RLS) problem

Given:

$A_0 x = b_0$ is unknown hypothetical exact compatible SLAE,
x_0 is its unknown hypothetical normal solution,
$Ax = b$ is the specified approximate (optional compatible) SLAE,
$\|A_0 - A\| \leqslant \mu, \|b_0 - b\| \leqslant \delta < \|b\|$, $\mu, \delta \geqslant 0$,
μ, δ is a priori known constants (not zero),
$\|\cdot\|$ is the Euclidean norm (for a vector) or the Frobenius norm (for a matrix).

Find:

A_1, x_1, b_1 such that $A_1 x_1 = b_1$, $\|A_1 - A\| \leqslant \mu, \|b_1 - b\| \leqslant \delta$,
$\|x_1\| \to \min$.

[1] A.N. Tikhonov, "On approximate systems of linear algebraic equations", Zh. Vychisl. Mat. Mat. Fiz., vol. 20, no. 6, pp. 1373–1383, 1980 [in Russian].

Solution of the RLS problem

Reduction RLS to problem $R(A, b, \mu, \delta)$:

Given: A, b, μ, δ

Find: x such that $\|b - Ax\| = \mu\|x\| + \delta$, $\|x\| \to \min$.

Recovery of the RLS solution

Let x^* be a solution of the $R(A, b, \mu, \delta)$. Then the solution of RLS problem has the form

$$x_1 = x^*,$$

$$b_1 = b - \frac{\delta}{\|b - Ax_1\|} \cdot (b - Ax_1),$$

$$A_1 = A + (b_1 - Ax_1) \cdot \frac{x_1^\top}{x_1^\top x_1}.$$

(Main tool): The minimal (by norm) solution of the SLAE with respect to the unknown matrix

$M(x, b)$ problem

Given: SLAE $Ax = b$ with unknown matrix A, $x \neq 0$.
Find: the matrix A with minimal norm.

The main lemma (by A.N. Tikhonov)

The solution of the problem $M(x, b)$ exists, is unique and has the form

$$\hat{A} = \frac{bx^\top}{x^\top x}.$$

At that

$$\|\hat{A}\| = \frac{\|b\|}{\|x\|}.$$

Evolution from the approximate SLAE to improper LP (1983, A.A. Vatolin, Ekaterinburg, IMM UB RAS, ... 2016, M.N. Khvostov, Moscow, CC RAS ...)

A simple problem of matrix correction LP $L(A,b,c)$: $Ax=b$, $x \geqslant 0$, $c^\top x \to \max$

$$\left\| \begin{bmatrix} H & -h \end{bmatrix} \right\| \to \min_{\text{The constraints of } (A+H)x=b+h,\ x\geqslant 0 \text{ are compatible}} .$$

Solution

Let $D = \begin{bmatrix} A & -b \end{bmatrix}^\top \begin{bmatrix} A & -b \end{bmatrix}$, $\varphi(z) = \frac{z^\top Dx}{z^\top z}$, $y \in \operatorname{Argmin}_{z\geqslant 0} \varphi(z)$.

The following assertions are true:

$\inf_{(A+H)x=b+h,\ x\geqslant 0} \left\| \begin{bmatrix} H & -h \end{bmatrix} \right\|^2 = \min_{z\geqslant 0} \varphi(z)$, Problem has a solution if and only if there is a vector y such that $y_{n+1} > 0$. At the same time $\begin{bmatrix} H^* & -h^{\ *} \end{bmatrix} = -\begin{bmatrix} A & -b \end{bmatrix} \frac{yy^\top}{y^\top y}$, $x^* = y_{n+1}^{-1} \cdot [y_1 \ldots y_n]^\top$ is the solution of system $(A+H^*)\,x = b + h^*, x \geqslant 0$.

A generalization of Tikhonov's lemma to a system of conjugate SLAE [2]

$M(x, b, u, v)$ problem

Given: $x, v \in \mathbb{R}^n$, $u, b \in \mathbb{R}^m$, $x, u \neq 0$.
Find: the matrix $A \in \mathbb{R}^{m \times n}$ with minimal norm, which is a solution of the system $Ax = b$, $u^\top A = v^\top$.

Solution

A solution of the problem $M(x, b, u, v)$ exists if and only if the condition $b^\top u = v^\top x = \alpha$. The solution is unique and has the form

$$\hat{A} = \frac{bx^\top}{x^\top x} + \frac{uv^\top}{u^\top u} - \alpha \cdot \frac{ux^\top}{x^\top x \cdot u^\top u}.$$

At that $\|\hat{A}\|^2 = \|b\|^2 \Big/ \|x\|^2 + \|v\|^2 \Big/ \|u\|^2 - \alpha^2 \Big/ \left(\|x\|^2 \cdot \|u\|^2\right)$.

[2] V.I. Erokhin, "Matrix Correction of a Dual Pair of Improper Linear Programming Problems", Comput. Math. Math. Phys., vol. 47, no. 4, pp. 564–578, 2007.

A simple problem of matrix correction of a dual pair of improper LP problems

Let $L(A,b,c)$: $Ax=b,\ x \geqslant 0,\ c^\top x \to \max$ and $L^*(A,b,c)$: $u^\top A x \geqslant c^\top,\ u^\top b \to \min$ be a primal and dual LP problem.

A simple matrix correction problem $S(A,b,c)$:

$$\|H\| \to \min_{L(A+H,b,c),\ L^*(A+H,b,c) \text{ are solvable}}$$

Solution (is generalized Tikhonov's lemma)

Problem $S(A,b,c)$ has a solution if and only if problem $\tilde{M}(A,x,b,u,c)$: $c^\top x = u^\top b = \gamma,\ d^\top x = 0,\ x \geqslant 0,\ d \geqslant 0,$
$\frac{\|b-Ax\|^2}{\|x\|^2} + \frac{\left\|c+d-A^\top u\right\|^2}{\|u\|^2} - \frac{\left(\gamma - u^\top A x\right)^2}{\|x\|^2 \cdot \|u\|^2} \to \min\limits_{x,d,u}$ is solvable.

Let x^*, u^* and d^* are the solution of the problem $\tilde{M}(A,x,b,u,c)$.
Then the solution of problem $S(A,b,c)$ has the form

$$H^* = \frac{(b-Ax^*){x^*}^\top}{{x^*}^\top x^*} + \frac{u^*(c+d^*-A^\top u^*)^\top}{{u^*}^\top u^*} - (c^\top x^* - {u^*}^\top A x^*) \cdot \frac{u^* {x^*}^\top}{{x^*}^\top x^* \cdot {u^*}^\top u^*}.$$

Evolution from matrix correction to approximate LP

Let $\mathbf{X}(A, b) \triangleq \{x \,|\, Ax = b, x \geqslant 0\}$ and $\mathbf{U}(A, c) \triangleq \left\{u \,\middle|\, u^\top A \geqslant c^\top\right\}$ are feasible sets of problems $L(A, b, c)$ and $L^*(A, b, c)$.

A simple approximate LP problem $P(A, b, c, \mu)$:

Given: A_0 is unknown hypothetical exact matrix, A is specified approximate matrix, $b = b_0$, $c = c_0$ are specified exact vectors, such that $\|A_0 - A\| \leqslant \mu$, $\mu > 0$ is a priory known constant, problems $L(A_0, b, c)$, $L^*(A_0, b, c)$ are solvable.

Find: A_1, x_1, u_1 such as $\|A_1 - A\| \leqslant \mu$,

$$\|x_1\|^2 + \|u_1\|^2 \to \min_{\mathbf{X}(A_1, b) \neq \varnothing, \mathbf{U}(A_1, c) \neq \varnothing}$$

Solution of the problem of $P(A, b, c, \mu)$

(is generalized Tikhonov's lemma)

The solution of the problem $P(A, b, c, \mu)$ exists and has the form

$$A_1 = A + H,$$

$$H = \frac{(b - Ax_1)x_1^\top}{x_1^\top x_1} + \frac{u_1(c + d_1 - A^\top u_1)^\top}{u_1^\top u_1} - \frac{(c^\top x_1 - u_1^\top A x_1) \cdot u_1 x_1^\top}{x_1^\top x_1 \cdot u_1^\top u_1},$$

where x_1, u_1, d_1 are solutions of the problem $R(A, b, c, \mu)$:

Find x, u, d such that $x, d \geqslant 0$, $d^\top x = 0$, $c^\top x = b^\top u = \gamma$,
$\alpha = \gamma - u^\top A x$, $\frac{\|b - Ax\|^2}{\|x\|^2} + \frac{\|c + d - A^\top u\|^2}{\|u\|^2} - \frac{\alpha^2}{\|x\|^2 \cdot \|u\|^2} = \mu^2$,
$\|x\|^2 + \|u\|^2 \to \min.$

Thanks for your attention!