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The example of best uniform approximation in multidimensional space

The harmonic mean of n positive numbers is defined by the following function:

$$f(x) = \frac{n}{\sum_{k=1}^{n} \frac{1}{x_k}}, \quad x \in \mathbb{R}^n_{++}.$$

This function can be continuously extended to $\mathbb{R}^n_{+}$ (if at least one x_k is zero, then $f(x) = 0$). Furthermore, f is postive homogeneous on this set: $f(\lambda x) = \lambda f(x)$ for all $\lambda > 0$.

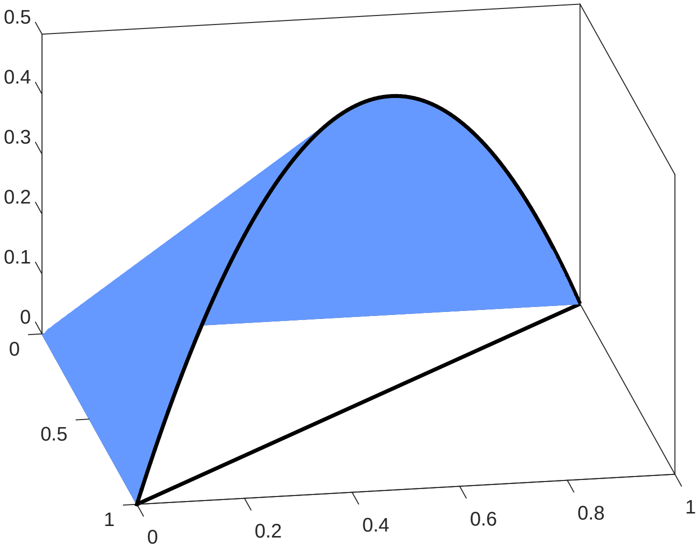

Рис. 1: The graph of the harmonic mean when $n = 2$

Consider the problem of unifrom approximation of function f by homogeneous function $g(c, x) = \langle c, x \rangle$ on a solid simplex $\Omega = \{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^n x_i \leqslant 1\}$:

$$\varphi(c) = \max_{x \in \Omega} |f(x) - g(c, x)| \to \inf_{c \in \mathbb{R}^n} .$$

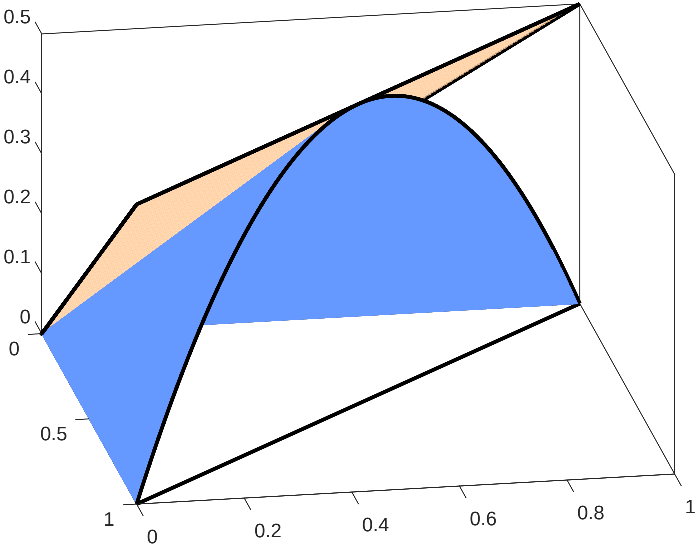

Рис. 2: Inequality between harmonic mean and arithmetic mean

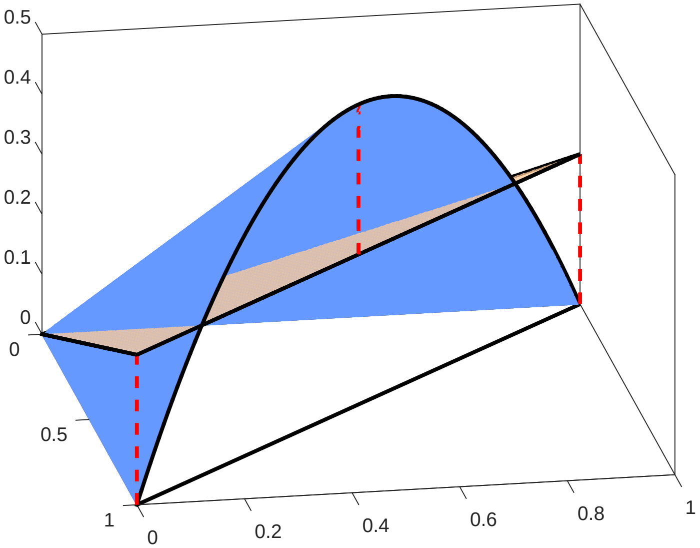

Рис. 3: Solution when $n = 2$

We show that vector $c_* = (\frac{1}{2n}, \ldots, \frac{1}{2n})$ is the unique solution of our approximation problem.

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Lemma 1

The following representation holds:

$$\varphi(c) = \max_{x \in \Lambda} |f(x) - g(c, x)|,$$

where $\Lambda = \{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^n x_i = 1\}$ *is standard simplex.*

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Lemma 2

The following equality holds:

$$\varphi(c_*) = \frac{1}{2n}.$$

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Theorem 1

Vector c_* *is the unique solution of considered approximation problem.*

Theorem 2 (strong uniqueness)

For any $c \in \mathbb{R}^n$ the following inequality holds:

$$\varphi(c_\alpha) - \varphi(c_*) \geqslant r||c - c_*||,$$

where $r = \frac{1}{\sqrt{4n^2 - 3n}}$.

Theorem 3

Constant r is unimprovable i.e. there exists c_α such that

$$\lim_{\alpha \to +0} \frac{\varphi(c_\alpha) - \varphi(c_*)}{||c_\alpha - c_*||} = r.$$