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Test Problems for Nonsmooth Unconstrained and Linearly Constrained Optimization

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Technical report No. 798

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Abstract

This report contains a description of subroutines which can be used for testing nonsmooth optimization codes. These subroutines can easily be obtained either by using the anonymous `ftp` (address `ftp://ftp.cs.cas.cz/pub/msdos/opt`, files `TEST06.FOR`, `TEST19.FOR`, `TEST22.FOR` or from the homepage `http://www.cs.cas.cz/~luksan /test.html`. Furthermore, all test problems contained in these subroutines are presented in the analytic form.

Keywords

nonsmooth optimization, test problems

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1 Introduction

This report describes three groups of subroutines which contain problems for testing nonsmooth optimization codes:

1. Unconstrained minimax optimization - subroutines `TIUD06`, `TAFU06`, `TAGU06`, `TAHD06`. We want to find a minimum of the minimax objective function
$$F(x) = \max_{1 \le k \le n_A} (f_k(x)), \quad x \in R^n,$$
or
$$F(x) = \max_{1 \le k \le n_A} |f_k(x)|, \quad x \in R^n,$$
where $f_k(x)$, $1 \le k \le n_A$, are twice continuously differentiable partial functions.

2. Unconstrained nonsmooth minimization - subroutines `TIUD19`, `TFFU19`, `TFGU19`, `TFHD19`. We want to find a minimum of the general nonsmooth objective function $F(x)$.

3. Linearly constrained minimax optimization - subroutines `TILD22`, `TAFU22`, `TAGU22`, `TAHD22`. We want to find a minimum of the minimax objective function
$$F(x) = \max_{1 \le k \le n_A} (f_k(x)), \quad x \in R^n,$$
or
$$F(x) = \max_{1 \le k \le n_A} |f_k(x)|, \quad x \in R^n,$$
where $f_k(x)$, $1 \le k \le n_A$, are twice continuously differentiable partial functions, subject to simple bounds and general linear constraints. Simple bounds are assumed in the form
$$\begin{aligned} x_i - \text{unbounded} \quad &, \quad I_i^x = 0, \\ x_i^l \le x_i \quad &, \quad I_i^x = 1, \\ x_i \le x_i^u \quad &, \quad I_i^x = 2, \\ x_i^l \le x_i \le x_i^u \quad &, \quad I_i^x = 3, \\ x_i = x_i^l = x_i^u \quad &, \quad I_i^x = 5, \end{aligned}$$
where $1 \le i \le n$. General linear constraints are assumed in the form
$$\begin{aligned} a_i^T x - \text{unbounded} \quad &, \quad I_i^c = 0, \\ c_i^l \le a_i^T x \quad &, \quad I_i^c = 1, \\ a_i^T x \le c_i^u \quad &, \quad I_i^c = 2, \\ c_i^l \le a_i^T x \le c_i^u \quad &, \quad I_i^c = 3, \\ a_i^T x = c_i^l = c_i^u \quad &, \quad I_i^c = 5, \end{aligned}$$
where $1 \le i \le n_c$ and n_c is a number of general linear constraints (I^x, I^c correspond to arays `IX, IC` in Section 3).

All subroutines are writen in the standard Fortran 77 language. Their names are derived from the following rule:

- The first letter is `T` - test subroutines.
- The second letter is either `I` - initiation, or `F` - objective function, or `A` - partial function.
- The third letter is either `U` - initiation of unconstrained problem, or `L` - initiation of linearly constrained problem, or `F` - computation of the function value, or `G` - computation of the function gradient, or `H` - computation of the function Hessian matrix.
- The fourth letter is either `U` - universal subroutine, or `D` - subroutine for dense problems.

The last two digits determine a given collection (numbering corresponds to the UFO system [13], which contains similar collections).

Initiation subroutines use the following parameters (array dimensions are given in parentheses):

`N`	input	number of variables.
`NA`	output	number of partial functions.
`NB`	output	number of box constraints.
`NC`	output	number of general linear constraints.
`X(N)`	output	vector of variables.
`IX(N)`	output	types of box constraints.
`XL(N)`	output	lower bounds in box constraints.
`XU(N)`	output	upper bounds in box constraints.
`IC(N)`	output	types of general linear constraints.
`CL(N)`	output	lower bounds in general linear constraints.
`CU(N)`	output	upper bounds in general linear constraints.
`FMIN`	output	lower bound for the objective function value.
`XMAX`	output	maximum stepsize.
`NEXT`	input	number of the problem selected.
`IEXT`	output	type of the objective function: -1 - maximum of values, 0 - maximum of absolute values.
`IERR`	output	error indicator: 0 - data are correct, 1 - `N` is too small.

Although `N` is an input parameter, it can be changed by the initiation subroutine when its value does not satisfy the required conditions. For example, most of the problems require `N` to be even or a multiple of a positive integer.

Evaluation subroutines use the following parameters (array dimensions are given in parentheses):

N	input	number of variables.
X(N)	input	vector of variables.
F	output	value of the objective function.
G(N)	output	gradient of the objective function.
H(N*(N+1)/2)	output	Hessian matrix of the objective function.
KA	input	index of the partial function selected.
FA	output	value of the partial function selected.
GA(N)	output	gradient of the partial function selected.
HA(N*(N+1)/2)	output	Hessian matrix of the partial function selected.
NEXT	input	number of the problem selected.

2 Test problems for unconstrained minimax optimization

Calling statements have the form

```
CALL TIUD06(N,NA,X,FMIN,XMAX,NEXT,IEXT,IERR),
CALL TAFU06(N,KA,X,FA,NEXT),
CALL TAGU06(N,KA,X,GA,NEXT),
CALL TAHD06(N,KA,X,HA,NEXT),
```

with the following significance:

- TIUD06 – determination of problem dimension N, NA and initiation of vector of variables X.
- TAFU06 – evaluation of the KA-th partial function value FA at point X.
- TAGU06 – evaluation of the KA-th partial function gradient GA at point X.
- TAHD06 – evaluation of the KA-th partial function Hessian matrix HA at point X.

We seek a minimum of a minimax objective function

$$F(x) = \max_{1 \le k \le n_A} (f_k(x)), \quad x \in R^n,$$

from the starting point $\bar{x}$. The description of individual problems follows. Table 2.1 contains problem dimensions and optimum function values.

No.	Problem	n	n_A	Optimum value
2.1	CB2	2	3	1.9522245
2.2	WF	2	3	0
2.3	SPIRAL	2	2	0
2.4	EVD52	3	6	3.5997193
2.5	Rosen-Suzuki	4	4	-44
2.6	Polak 6	4	4	-44
2.7	PBC3	3	21	$0.42021427 \cdot 10^{-2}$
2.8	Bard	3	15	$0.50816327 \cdot 10^{-1}$
2.9	Kowalik-Osborne	4	11	$0.80843684 \cdot 10^{-2}$
2.10	Davidon 2	4	20	115.70644
2.11	OET5	4	21	$0.26359735 \cdot 10^{-2}$
2.12	OET6	4	21	$0.20160753 \cdot 10^{-2}$
2.13	GAMMA	4	61	$0.12041887 \cdot 10^{-6}$
2.14	EXP	5	21	$0.12237125 \cdot 10^{-3}$
2.15	PBC1	5	30	$0.22340496 \cdot 10^{-1}$
2.16	EVD61	6	51	$0.34904926 \cdot 10^{-1}$
2.17	Transformer	6	11	0.19729063
2.18	Filter	9	41	$0.61852848 \cdot 10^{-2}$
2.19	Wong 1	7	5	680.63006
2.20	Wong 2	10	9	24.306209
2.21	Wong 3	20	18	133.72828
2.22	Polak 2	10	2	54.598150
2.23	Polak 3	11	10	261.08258
2.24	Watson	20	31	$0.14743027 \cdot 10^{-7}$
2.25	Osborne 2	11	65	$0.48027401 \cdot 10^{-1}$

Table 2.1

Problem 2.1 CB2 [5].

$$
\begin{aligned}
F(x) &= \max_{1 \le i \le 3} f_i(x), \\
f_1(x) &= x_1^2 + x_2^4, \\
f_2(x) &= (2 - x_1)^2 + (2 - x_2)^2, \\
f_3(x) &= 2 \exp(x_2 - x_1), \\
\overline{x}_1 &= 2, \qquad \overline{x}_2 \;=\; 2.
\end{aligned}
$$

Problem 2.2 WF [16].

$$
\begin{aligned}
F(x) &= \max_{1 \le i \le 3} f_i(x), \\
f_1(x) &= \frac{1}{2}\left(x_1 + \frac{10x_1}{x_1 + 0.1} + 2x_2^2\right),
\end{aligned}
$$

$$
\begin{aligned}
f_2(x) &= \frac{1}{2}\left(-x_1 + \frac{10x_1}{x_1+0.1} + 2x_2^2\right), \\
f_3(x) &= \frac{1}{2}\left(x_1 - \frac{10x_1}{x_1+0.1} + 2x_2^2\right), \\
\overline{x}_1 &= 3, \qquad \overline{x}_2 \;=\; 1.
\end{aligned}
$$

Problem 2.3 SPIRAL [16].

$$
\begin{aligned}
F(x) &= \max(f_1(x), f_2(x)), \\
f_1(x) &= \left(x_1 - \sqrt{x_1^2+x_2^2}\cos\sqrt{x_1^2+x_2^2}\right)^2 + 0.005(x_1^2+x_2^2), \\
f_2(x) &= \left(x_2 - \sqrt{x_1^2+x_2^2}\sin\sqrt{x_1^2+x_2^2}\right)^2 + 0.005(x_1^2+x_2^2), \\
\overline{x}_1 &= 1.41831, \qquad \overline{x}_2 \;=\; -4.79462.
\end{aligned}
$$

Problem 2.4 EVD52 [7].

$$
\begin{aligned}
F(x) &= \max_{1\le i\le 6} f_i(x), \\
f_1(x) &= x_1^2 + x_2^2 + x_3^2 - 1, \\
f_2(x) &= x_1^2 + x_2^2 + (x_3-2)^2, \\
f_3(x) &= x_1 + x_2 + x_3 - 1, \\
f_4(x) &= x_1 + x_2 - x_3 + 1, \\
f_5(x) &= 2x_1^3 + 6x_2^2 + 2(5x_3 - x_1 + 1)^2, \\
f_6(x) &= x_1^2 - 9x_3, \\
\overline{x}_i &= 1, \quad 1 \le i \le 3.
\end{aligned}
$$

Problem 2.5 Rosen-Suzuki [20].

$$
\begin{aligned}
F(x) &= \max_{1\le i\le 4} f_i(x), \\
f_1(x) &= x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4, \\
f_2(x) &= f_1(x) + 10(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 - x_2 + x_3 - x_4 - 8), \\
f_3(x) &= f_1(x) + 10(x_1^2 + 2x_2^2 + x_3^2 + 2x_4^2 - x_1 - x_4 - 10), \\
f_4(x) &= f_1(x) + 10(2x_1^2 + x_2^2 + x_3^2 + 2x_1 - x_2 - x_4 - 5), \\
\overline{x}_1 &= 0, \qquad \overline{x}_2 \;=\; 0, \qquad \overline{x}_3 \;=\; 0, \qquad \overline{x}_4 \;=\; 0.
\end{aligned}
$$

Problem 2.6 Polak 6 [17].

$$
\begin{aligned}
F(x) &= \max_{1\le i\le 4} f_i(x), \\
f_1(x) &= (x_1-(x_4+1)^4)^2+(x_2-(x_1-(x_4+1)^4)^4)^2+2x_3^2+x_4^2- \\
&\quad -5(x_1-(x_4+1)^4)-5(x_2-(x_1-(x_4+1)^4)^4)-21x_3+7x_4, \\
f_2(x) &= f_1(x)+10((x_1-(x_4+1)^4)^2+(x_2-(x_1-(x_4+1)^4)^4)^2+x_3^2+x_4^2+ \\
&\quad +(x_1-(x_4+1)^4)-(x_2-(x_1-(x_4+1)^4)^4)+x_3-x_4-8), \\
f_3(x) &= f_1(x)+10((x_1-(x_4+1)^4)^2+2(x_2-(x_1-(x_4+1)^4)^4)^2+x_3^2+2x_4^2- \\
&\quad -(x_1-(x_4+1)^4)-x_4-10), \\
f_4(x) &= f_1(x)+10((x_1-(x_4+1)^4)^2+(x_2-(x_1-(x_4+1)^4)^4)^2+x_3^2+ \\
&\quad +2(x_1-(x_4+1)^4)-(x_2-(x_1-(x_4+1)^4)^4)-x_4-5), \\
\overline{x}_i &= 0, \quad 1\le i\le 4.
\end{aligned}
$$

Problem 2.7 PBC3 [18].

$$
\begin{aligned}
F(x) &= \max_{1\le i\le 21} |f_i(x)|, \\
f_i(x) &= \frac{x_3}{x_2}\exp(-t_i x_1)\sin(t_i x_2)-y_i, \\
y_i &= \frac{3}{20}e^{-t_i}+\frac{1}{52}e^{-5t_i}-\frac{1}{65}e^{-2t_i}\left(3\sin 2t_i+11\cos 2t_i\right), \\
t_i &= 10(i-1)/20, \quad 1\le i\le 21, \\
\overline{x}_i &= 1, \qquad 1\le i\le 3.
\end{aligned}
$$

Problem 2.8 Bard [21].

$$
\begin{aligned}
F(x) &= \max_{1\le i\le 15} |f_i(x)|, \\
f_i(x) &= x_1+\frac{u_i}{v_i x_2+w_i x_3}-y_i, \\
u_i &= i, \quad v_i = 16-i, \quad w_i = \min(u_i,v_i), \quad 1\le i\le 15, \\
\overline{x}_i &= 1, \quad 1\le i\le 3.
\end{aligned}
$$

Problem 2.9 Kowalik-Osborne [21].

$$
\begin{aligned}
F(x) &= \max_{1\le i\le 11} |f_i(x)|, \\
f_i(x) &= \frac{x_1(u_i^2+x_2u_i)}{u_i^2+x_3u_i+x_4}-y_i,
\end{aligned}
$$

i	y_i	u_i
1	0.1957	4.0000
2	0.1947	2.0000
3	0.1735	1.0000
4	0.1600	0.5000
5	0.0844	0.2500
6	0.0627	0.1670
7	0.0456	0.1250
8	0.0342	0.1000
9	0.0323	0.0833
10	0.0235	0.0714
11	0.0246	0.0625

$$\overline{x}_1 = 0.250, \quad \overline{x}_2 = 0.390, \quad \overline{x}_3 = 0.415 \quad \overline{x}_4 = 0.390.$$

Problem 2.10 Davidon 2 [21].

$$\begin{aligned}
F(x) &= \max_{1\le i\le 20} |f_i(x)|, \\
f_i(x) &= (x_1 + x_2 t_i - \exp(t_i))^2 + (x_3 + x_4 \sin(t_i) - \cos(t_i))^2, \\
t_i &= 0.2i, \qquad 1 \le i \le 20, \\
\overline{x}_1 &= 25, \quad \overline{x}_2 = 5, \quad \overline{x}_3 = -5, \quad \overline{x}_4 = -1.
\end{aligned}$$

Problem 2.11 OET5 [22].

$$\begin{aligned}
F(x) &= \max_{1\le i\le 21} |f_i(x)|, \\
f_i(x) &= x_4 - (x_1 t_i^2 + x_2 t_i + x_3)^2 - \sqrt{t_i}, \\
t_i &= 0.25 + 0.75(i-1)/20, \quad 1 \le i \le 21, \\
\overline{x}_i &= 1.0, \quad 1 \le i \le 4.
\end{aligned}$$

Problem 2.12 OET6 [22].

$$\begin{aligned}
F(x) &= \max_{1\le i\le 21} |f_i(x)|, \\
f_i(x) &= x_1 e^{x_3 t_i} + x_2 e^{x_4 t_i} - \frac{1}{1+t_i}, \\
t_i &= -0.5 + (i-1)/20, \quad 1 \le i \le 21, \\
\overline{x}_1 &= 1.0, \qquad \overline{x}_2 = 1.0, \qquad \overline{x}_3 = -3.0, \\
\overline{x}_4 &= -1.0.
\end{aligned}$$

Problem 2.13 GAMMA.

$$
\begin{aligned}
F(x) &= \max_{1\le i\le 61} |f_i(x)|, \\
f_i(x) &= \frac{x_1\left(t_i + x_2 + \frac{1}{x_3 t_i + x_4}\right)^{\left(t_i+\frac{1}{2}\right)}}{\Gamma(t_i+1)\exp(t_i)} - 1,
\end{aligned}
$$

i	t_i	i	t_i	i	t_i
1	1.000	22	2.200	43	13.00
2	1.010	23	2.300	44	15.00
3	1.020	24	2.500	45	17.50
4	1.030	25	2.750	46	20.00
5	1.050	26	3.000	47	22.50
6	1.075	27	3.250	48	25.00
7	1.100	28	3.500	49	30.00
8	1.125	29	4.000	50	35.00
9	1.150	30	4.500	51	40.00
10	1.200	31	5.000	52	50.00
11	1.250	32	5.500	53	60.00
12	1.300	33	6.000	54	70.00
13	1.350	34	6.500	55	80.00
14	1.400	35	7.000	56	100.0
15	1.500	36	7.500	57	150.0
16	1.600	37	8.000	58	200.0
17	1.700	38	8.500	59	300.0
18	1.800	39	9.000	60	500.0
19	1.900	40	10.00	61	10^5
20	2.000	41	11.00		
21	2.100	42	12.00		

$$
\overline{x}_1 = 1, \qquad \overline{x}_2 = 1, \qquad \overline{x}_3 = 10, \qquad \overline{x}_4 = 1.
$$

Problem 2.14 EXP [9].

$$
\begin{aligned}
F(x) &= \max_{1\le i\le 21} f_i(x), \\
f_i(x) &= \frac{x_1 + x_2 t_i}{1 + x_3 t_i + x_4 t_i^2 + x_5 t_i^3} - \exp(t_i), \\
t_i &= -1 + (i-1)/10, \quad 1 \le i \le 21, \\
\overline{x}_1 &= 0.5, \qquad \overline{x}_2 = 0, \qquad \overline{x}_3 = 0, \qquad \overline{x}_4 = 0, \qquad \overline{x}_5 = 0.
\end{aligned}
$$

Problem 2.15 PBC1 [18].

$$\begin{aligned}
F(x) &= \max_{1\le i\le 30} |f_i(x)|,\\
f_i(x) &= \frac{x_1 + x_2 t_i + x_3 t_i^2}{1 + x_4 t_i + x_5 t_i^2} - \frac{\sqrt{(8t_i-1)^2+1}\,\arctan(8t_i)}{8t_i},\\
t_i &= -1 + 2(i-1)/29, \quad 1 \le i \le 30,\\
\overline{x}_1 &= 0, \quad \overline{x}_2 = -1, \quad \overline{x}_3 = 10, \quad \overline{x}_4 = 1, \quad \overline{x}_5 = 10.
\end{aligned}$$

Problem 2.16 EVD61 [7].

$$\begin{aligned}
F(x) &= \max_{1\le i\le 51} |f_i(x)|,\\
f_i(x) &= x_1 \exp(-x_2 t_i)\cos(x_3 t_i + x_4) + x_5 \exp(-x_6 t_i) - y_i,\\
y_i &= 0.5e^{-t_i} - e^{-2t_i} + 0.5e^{-3t_i} + 1.5e^{-1.5t_i}\sin 7t_i + e^{-2.5t_i}\sin 5t_i,\\
t_i &= 0.1(i-1), \qquad 1 \le i \le 51,\\
\overline{x}_1 &= 2, \qquad \overline{x}_2 = 2, \qquad \overline{x}_3 = 7,\\
\overline{x}_4 &= 0, \qquad \overline{x}_5 = -2, \qquad \overline{x}_0 = 1.
\end{aligned}$$

Problem 2.17 Transformer [2].

$$\begin{aligned}
F(x) &= \max_{1\le i\le 11} f_i(x),\\
f_i(x) &= \left| 1 - 2\frac{v_1(x,t_i)}{w_1(x,t_i) + v_1(x,t_i)} \right|,
\end{aligned}$$

where $v_1(x,t_i)$ and $w_1(x,t_i)$ are complex numbers obtained recursively so that $v_4(x,t_i) = 1$, $w_4(x,t_i) = 10$ and

$$\begin{aligned}
v_k(x,t_i) &= \cos(\vartheta_i x_{2k-1}) v_{k+1}(x,t_i) + j\sin(\vartheta_i x_{2k-1}) \frac{1}{x_{2k}} w_{k+1}(x,t_i),\\
w_k(x,t_i) &= \cos(\vartheta_i x_{2k-1}) w_{k+1}(x,t_i) + j\sin(\vartheta_i x_{2k-1}) x_{2k} v_{k+1}(x,t_i)
\end{aligned}$$

for $k = 3, 2, 1$. Here $j = \sqrt{-1}$ is the imaginary unit and

$$\begin{aligned}
\vartheta_i &= (\pi/2)\, t_i, \quad 1 \le i \le 11,\\
t_1 &= 0.5, \quad t_2 = 0.6, \quad t_3 = 0.7, \quad t_4 = 0.77,\\
t_5 &= 0.9, \quad t_6 = 1.0, \quad t_7 = 1.1, \quad t_8 = 1.23,\\
t_9 &= 1.3, \quad t_{10} = 1.4, \quad t_{11} = 1.5,\\
\overline{x}_1 &= 0.8, \quad \overline{x}_2 = 1.5, \quad \overline{x}_3 = 1.2,\\
\overline{x}_4 &= 3.0, \quad \overline{x}_5 = 0.8, \quad \overline{x}_6 = 6.0.
\end{aligned}$$

Problem 2.18 Filter [4].

$$
\begin{aligned}
F(x) &= \max_{1\le i\le 41} |f_i(x)|, \\
f_i(x) &= \left(\frac{(x_1+(1+x_2)\cos\vartheta_i)^2+((1-x_2)\sin\vartheta_i)^2}{(x_3+(1+x_4)\cos\vartheta_i)^2+((1-x_4)\sin\vartheta_i)^2}\right)^{\frac{1}{2}} \cdot \\
&\quad \left(\frac{(x_5+(1+x_6)\cos\vartheta_i)^2+((1-x_6)\sin\vartheta_i)^2}{(x_7+(1+x_8)\cos\vartheta_i)^2+((1-x_8)\sin\vartheta_i)^2}\right)^{\frac{1}{2}} x_9 - y_i, \\
y_i &= |1-2t_i|, \quad \vartheta_i \;=\; \pi t_i \\
t_i &= 0.01(i-1), \quad 1\le i\le 6, \\
t_i &= 0.07+0.03(i-7), \quad 7\le i\le 20, \quad t_{21}=0.50, \\
t_i &= 0.54+0.03(i-22), \quad 22\le i\le 35, \\
t_i &= 0.95+0.01(i-36), \quad 36\le i\le 41, \\
\overline{x}_1 &= 0.00, \qquad \overline{x}_2 \;=\; 1.00, \qquad \overline{x}_3 \;=\; 0.00, \qquad \overline{x}_4 \;=\; -0.15, \\
\overline{x}_5 &= 0.00, \qquad \overline{x}_6 \;=\; -0.68, \qquad \overline{x}_7 \;=\; 0.00, \qquad \overline{x}_8 \;=\; -0.72, \\
\overline{x}_9 &= 0.37.
\end{aligned}
$$

Problem 2.19 Wong 1 [1].

$$
\begin{aligned}
F(x) &= \max_{1\le i\le 5} f_i(x), \\
f_1(x) &= (x_1-10)^2+5(x_2-12)^2+x_3^4+3(x_4-11)^2+10x_5^6+7x_6^2+x_7^4- \\
&\quad -4x_6x_7-10x_6-8x_7, \\
f_2(x) &= f_1(x)+10(2x_1^2+3x_2^4+x_3+4x_4^2+5x_5-127), \\
f_3(x) &= f_1(x)+10(7x_1+3x_2+10x_3^2+x_4-x_5-282), \\
f_4(x) &= f_1(x)+10(23x_1+x_2^2+6x_6^2-8x_7-196), \\
f_5(x) &= f_1(x)+10(4x_1^2+x_2^2-3x_1x_2+2x_3^2+5x_6-11x_7), \\
\overline{x}_1 &= 1, \quad \overline{x}_2 \;=\; 2, \quad \overline{x}_3 \;=\; 0, \quad \overline{x}_4 \;=\; 4, \quad \overline{x}_5 \;=\; 0, \quad \overline{x}_6 \;=\; 1, \quad \overline{x}_7 \;=\; 1.
\end{aligned}
$$

Problem 2.20 Wong 2 [1].

$$
\begin{aligned}
F(x) &= \max_{1\le i\le 9} f_i(x), \\
f_1(x) &= x_1^2+x_2^2+x_1x_2-14x_1-16x_2+(x_3-10)^2+4(x_4-5)^2+(x_5-3)^2+ \\
&\quad +2(x_6-1)^2+5x_7^2+7(x_8-11)^2+2(x_9-10)^2+(x_{10}-7)^2+45, \\
f_2(x) &= f_1(x)+10(3(x_1-2)^2+4(x_2-3)^2+2x_3^2-7x_4-120), \\
f_3(x) &= f_1(x)+10(5x_1^2+8x_2+(x_3-6)^2-2x_4-40), \\
f_4(x) &= f_1(x)+10(0.5(x_1-8)^2+2(x_2-4)^2+3x_5^2-x_6-30), \\
f_5(x) &= f_1(x)+10(x_1^2+2(x_2-2)^2-2x_1x_2+14x_5-6x_6),
\end{aligned}
$$

$$
\begin{aligned}
f_6(x) &= f_1(x) + 10(4x_1 + 5x_2 - 3x_7 + 9x_8 - 105),\\
f_7(x) &= f_1(x) + 10(10x_1 - 8x_2 - 17x_7 + 2x_8),\\
f_8(x) &= f_1(x) + 10(-3x_1 + 6x_2 + 12(x_9 - 8)^2 - 7x_{10}),\\
f_9(x) &= f_1(x) + 10(-8x_1 + 2x_2 + 5x_9 - 2x_{10} - 12),\\
\overline{x}_1 &= 2, \quad \overline{x}_2 = 3, \quad \overline{x}_3 = 5, \quad \overline{x}_4 = 5, \quad \overline{x}_5 = 1, \quad \overline{x}_6 = 2,\\
\overline{x}_7 &= 7, \quad \overline{x}_8 = 3, \quad \overline{x}_9 = 6, \quad \overline{x}_{10} = 10.
\end{aligned}
$$

Problem 2.21 Wong 3 [1].

$$
\begin{aligned}
F(x) &= \max_{1\le i\le 18} f_i(x),\\
f_1(x) &= x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3 - 10)^2 + 4(x_4 - 5)^2 + (x_5 - 3)^2 +\\
&\quad +2(x_6 - 1)^2 + 5x_7^2 + 7(x_8 - 11)^2 + 2(x_9 - 10)^2 + (x_{10} - 7)^2 + (x_{11} - 9)^2 +\\
&\quad +10(x_{12} - 1)^2 + 5(x_{13} - 7)^2 + 4(x_{14} - 14)^2 + 27(x_{15} - 1)^2 + x_{16}^4 + (x_{17} - 2)^2 +\\
&\quad +13(x_{18} - 2)^2 + (x_{19} - 3)^2 + x_{20}^2 + 95,\\
f_2(x) &= f_1(x) + 10(3(x_1 - 2)^2 + 4(x_2 - 3)^2 + 2x_3^2 - 7x_4 - 120),\\
f_3(x) &= f_1(x) + 10(5x_1^2 + 8x_2 + (x_3 - 6)^2 - 2x_4 - 40),\\
f_4(x) &= f_1(x) + 10(0.5(x_1 - 8)^2 + 2(x_2 - 4)^2 + 3x_5^2 - x_6 - 30),\\
f_5(x) &= f_1(x) + 10(x_1^2 + 2(x_2 - 2)^2 - 2x_1x_2 + 14x_5 - 6x_6),\\
f_6(x) &= f_1(x) + 10(4x_1 + 5x_2 - 3x_7 + 9x_8 - 105),\\
f_7(x) &= f_1(x) + 10(10x_1 - 8x_2 - 17x_7 + 2x_8),\\
f_8(x) &= f_1(x) + 10(-3x_1 + 6x_2 + 12(x_9 - 8)^2 - 7x_{10}),\\
f_9(x) &= f_1(x) + 10(-8x_1 + 2x_2 + 5x_9 - 2x_{10} - 12),\\
f_{10}(x) &= f_1(x) + 10(x_1 + x_2 + 4x_{11} - 21x_{12}),\\
f_{11}(x) &= f_1(x) + 10(x_1^2 + 5x_{11} - 8x_{12} - 28),\\
f_{12}(x) &= f_1(x) + 10(4x_1 + 9x_2 + 5x_{13}^2 - 9x_{14} - 87),\\
f_{13}(x) &= f_1(x) + 10(3x_1 + 4x_2 + 3(x_{13} - 6)^2 - 14x_{14} - 10),\\
f_{14}(x) &= f_1(x) + 10(14x_1^2 + 35x_{15} - 79x_{16} - 92),\\
f_{15}(x) &= f_1(x) + 10(15x_2^2 + 11x_{15} - 61x_{16} - 54),\\
f_{16}(x) &= f_1(x) + 10(5x_1^2 + 2x_2 + 9x_{17}^4 - x_{18} - 68),\\
f_{17}(x) &= f_1(x) + 10(x_1^2 - x_2 + 19x_{19} - 20x_{20} + 19),\\
f_{18}(x) &= f_1(x) + 10(7x_1^2 + 5x_2^2 + x_{19}^2 - 30x_{20}),\\
\overline{x}_1 &= 2, \quad \overline{x}_2 = 3, \quad \overline{x}_3 = 5, \quad \overline{x}_4 = 5, \quad \overline{x}_5 = 1, \quad \overline{x}_6 = 2, \quad \overline{x}_7 = 7,\\
\overline{x}_8 &= 3, \quad \overline{x}_9 = 6, \quad \overline{x}_{10} = 10, \quad \overline{x}_{11} = 2, \quad \overline{x}_{12} = 2, \quad \overline{x}_{13} = 6, \quad \overline{x}_{14} = 15,\\
\overline{x}_{15} &= 1, \quad \overline{x}_{16} = 2, \quad \overline{x}_{17} = 1, \quad \overline{x}_{18} = 2, \quad \overline{x}_{19} = 1, \quad \overline{x}_{20} = 3.
\end{aligned}
$$

Problem 2.22 Polak 2 [17]

$$\begin{aligned}
F(x) &= \max\left\{f(x+2e_2), f(x-2e_2)\right\}, \\
f(x) &= \exp(10^{-8}x_1^2 + x_2^2 + x_3^2 + 4x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2 + x_9^2 + x_{10}^2), \\
e_2 &= \text{second column of the unit matrix}, \\
\overline{x}_1 &= 100, \quad \overline{x}_i = 0.1, \quad 2 \le i \le 10.
\end{aligned}$$

Problem 2.23 Polak 3 [17].

$$\begin{aligned}
F(x) &= \max_{1\le i\le 10} f_i(x), \\
f_i(x) &= \sum_{j=0}^{10} \frac{1}{i+j} \exp\left((x_{j+1} - \sin(i-1+2j))^2\right), \\
\overline{x}_i &= 1, \quad 1 \le i \le 11.
\end{aligned}$$

Problem 2.24 Watson [21].

$$\begin{aligned}
F(x) &= \max_{1\le i\le 31} |f_i(x)|, \\
f_1(x) &= x_1, \\
f_2(x) &= x_2 - x_1^2 - 1, \\
f_i(x) &= \sum_{j=2}^{n} (j-1)x_j \left(\frac{i-2}{29}\right)^{j-2} - \left[\sum_{j=1}^{n} x_j \left(\frac{i-2}{29}\right)^{j-1}\right]^2, \quad 3 \le j \le 31, \\
\overline{x}_i &= 0, \qquad 1 \le i \le 20.
\end{aligned}$$

Problem 2.25 Osborne 2 [21].

$$\begin{aligned}
F(x) &= \max_{1\le i\le 65} |f_i(x)|, \\
f_i(x) &= y_i - x_1\exp(-x_5t_i) - x_2\exp(-x_6(t_i-x_9)^2) - x_3\exp(-x_7(t_i-x_{10})^2) - \\
&\quad - x_4\exp(-x_8(t_i-x_{11})^2), \\
t_i &= 0.1(i-1), \qquad 1 \le i \le 65.
\end{aligned}$$

i	y_i	i	y_i	i	y_i
1	1.366	23	0.694	45	0.672
2	1.191	24	0.644	46	0.708
3	1.112	25	0.624	47	0.633
4	1.013	26	0.661	48	0.668
5	0.991	27	0.612	49	0.645
6	0.885	28	0.558	50	0.632
7	0.831	29	0.533	51	0.591
8	0.847	30	0.495	52	0.559
9	0.786	31	0.500	53	0.597
10	0.725	32	0.423	54	0.625
11	0.746	33	0.395	55	0.739
12	0.679	34	0.375	56	0.710
13	0.608	35	0.372	57	0.729
14	0.655	36	0.391	58	0.720
15	0.616	37	0.396	59	0.636
16	0.606	38	0.405	60	0.581
17	0.602	39	0.428	61	0.428
18	0.626	40	0.429	62	0.292
19	0.651	41	0.523	63	0.162
20	0.724	42	0.562	64	0.098
21	0.649	43	0.607	65	0.054
22	0.649	44	0.653		

$$
\begin{aligned}
\overline{x}_1 &= 1.30, & \overline{x}_2 &= 0.65, & \overline{x}_3 &= 0.65, & \overline{x}_4 &= 0.70, \\
\overline{x}_5 &= 0.60, & \overline{x}_6 &= 3.00, & \overline{x}_7 &= 5.00, & \overline{x}_8 &= 7.00, \\
\overline{x}_9 &= 2.00, & \overline{x}_{10} &= 4.50, & \overline{x}_{11} &= 5.50.
\end{aligned}
$$

3 Test problems for general nonsmooth unconstrained optimization

Calling statements have the form

```
CALL TIUD19(N,X,FMIN,XMAX,NEXT,IERR),
CALL TFFU19(N,X,F,NEXT),
CALL TFGU19(N,X,G,NEXT),
CALL TFHD19(N,X,H,NEXT),
```

with the following significance:

`TIUD19` – initiation of vector of variables `N, X`.

`TFFU19` – evaluation of the objective function value `F` at point `X`.
`TFGU19` – evaluation of the objective function gradient `G` at point `X`.
`TFHD19` – evaluation of the objective function Hessian matrix `H` at point `X`.

We seek a local minimum of the objective function $F(x)$ from the starting point $\bar{x}$. The description of individual problems follows. Table 3.1 contains problem dimensions and optimum function values.

No.	Problem	n	Optimum value
3.1	Rosenbrock	2	0
3.2	Crescent	2	0
3.3	CB2	2	1.9522245
3.4	CB3	2	2
3.5	DEM	2	-3
3.6	QL	2	7.20
3.7	LQ	2	-1.4142136
3.8	Mifflin 1	2	-1
3.9	Mifflin 2	2	-1
3.10	Wolfe	2	-8
3.11	Rosen-Suzuki	4	-44
3.12	Shor	5	22.600162
3.13	Colville 1	5	-32.348679
3.14	HS78	5	-2.9197004
3.15	El-Attar	6	0.5598131
3.16	Maxquad	10	-0.8414083
3.17	Gill	10	9.7857721
3.18	Steiner 2	12	16.703838
3.19	Maxq	20	0
3.20	Maxl	20	0
3.21	TR48	48	-638565.0
3.22	Goffin	50	0
3.23	MXHILB	50	0
3.24	L1HILB	50	0
3.25	Shell Dual	15	32.348679

Table 3.1

Problem 3.1 Rosenbrock [15].

$$\begin{aligned} F(x) &= 100(x_2 - x_1^2)^2 + (1 - x_1)^2, \\ \overline{x}_1 &= -1.2, \qquad \overline{x}_2 = 1.0. \end{aligned}$$

Problem 3.2 Crescent [15].

$$\begin{aligned} F(x) &= \max\left\{x_1^2 + (x_2 - 1)^2 + x_2 - 1,\ -x_1^2 - (x_2 - 1)^2 + x_2 + 1\right\}, \\ \overline{x}_1 &= -1.5, \qquad \overline{x}_2 = 2.0. \end{aligned}$$

Problem 3.3 CB2 [15].

$$\begin{aligned} F(x) &= \max\left\{x_1^2+x_2^4, (2-x_1)^2+(2-x_2)^2,\ 2e^{-x_1+x_2}\right\}, \\ \overline{x}_1 &= 1.0, \qquad \overline{x}_2 \ = \ -0.1. \end{aligned}$$

Problem 3.4 CB3 [15].

$$\begin{aligned} F(x) &= \max\left\{x_1^4+x_2^2,\ (2-x_1)^2+(2-x_2)^2,\ 2e^{-x_1+x_2}\right\}, \\ \overline{x}_1 &= 2, \qquad \overline{x}_2 \ = \ 2. \end{aligned}$$

Problem 3.5 DEM [15].

$$\begin{aligned} F(x) &= \max\left\{5x_1+x_2, -5x_1+x_2,\ x_1^2+x_2^2+4x_2\right\}, \\ \overline{x}_1 &= 1, \qquad \overline{x}_2 \ = \ 1. \end{aligned}$$

Problem 3.6 QL [15].

$$\begin{aligned} F(x) &= \max_{1\le i\le 3} f_i(x), \\ f_1(x) &= x_1^2+x_2^2, \\ f_2(x) &= x_1^2+x_2^2+10(-4x_1-x_2+4), \\ f_3(x) &= x_1^2+x_2^2+10(-x_1-2x_2+6), \\ \overline{x}_1 &= -1, \qquad \overline{x}_2 \ = \ 5. \end{aligned}$$

Problem 3.7 LQ [15].

$$\begin{aligned} F(x) &= \max\left\{-x_1-x_2,\ -x_1-x_2+(x_1^2+x_2^2-1)\right\}, \\ \overline{x}_1 &= -0.5, \qquad \overline{x}_2 \ = \ -0.5. \end{aligned}$$

Problem 3.8 Mifflin 1 [15].

$$\begin{aligned} F(x) &= -x_1+20\max\left\{x_1^2+x_2^2-1,\ 0\right\}, \\ \overline{x}_1 &= 0.8, \qquad \overline{x}_2 \ = \ 0.6. \end{aligned}$$

Problem 3.9 Mifflin 2 [15].

$$\begin{aligned} F(x) &= -x_1+2(x_1^2+x_2^2-1)+1.75|x_1^2+x_2^2-1|, \\ \overline{x}_1 &= -1, \qquad \overline{x}_2 \ = \ -1. \end{aligned}$$

Problem 3.10 Wolfe [23].

$$\begin{aligned} F(x) &= f_1(x)\,, \quad x_1 \geq |x_2|, \\ F(x) &= f_2(x)\,, \quad 0 < x_1 < |x_2|, \\ F(x) &= f_3(x)\,, \quad x_1 \leq 0, \\ \\ f_1(x) &= 5\sqrt{(9x_1^2 + 16x_2^2)}, \\ f_2(x) &= 9x_1 + 16|x_2|, \\ f_3(x) &= 9x_1 + 16|x_2| - x_1^9, \\ \\ \overline{x}_1 &= 3, \qquad \overline{x}_2 \;=\; 2. \end{aligned}$$

Problem 3.11 Rosen-Suzuki [15].

$$\begin{aligned} F(x) &= \max\left\{f_1(x), f_1(x) + 10f_2(x), f_1(x) + 10f_3(x), f_1(x) + 10f_4(x)\right\}, \\ f_1(x) &= x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4, \\ f_2(x) &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 - x_2 + x_3 - x_4 - 8, \\ f_3(x) &= x_1^2 + 2x_2^2 + x_3^2 + 2x_4^2 - x_1 - x_4 - 10, \\ f_4(x) &= x_1^2 + x_2^2 + x_3^2 + 2x_1 - x_2 - x_4 - 5, \\ \overline{x}_i &= 0, \quad 1 \leq i \leq 4. \end{aligned}$$

Problem 3.12 Shor [15].

$$\begin{aligned} F(x) &= \max_{1\leq i\leq 10}\left\{b_i \sum_{j=1}^{5}(x_j - a_{ij})^2\right\}, \\ A &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 3 \\ 1 & 2 & 1 & 1 & 2 \\ 1 & 4 & 1 & 2 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 \end{bmatrix}, \qquad b \;=\; \begin{bmatrix} 1 \\ 5 \\ 10 \\ 2 \\ 4 \\ 3 \\ 1.7 \\ 2.5 \\ 6 \\ 3.5 \end{bmatrix}, \\ \overline{x}_1 &= 0, \qquad \overline{x}_2 \;=\; 0, \qquad \overline{x}_3 \;=\; 0, \qquad \overline{x}_4 \;=\; 0, \qquad \overline{x}_5 \;=\; 1. \end{aligned}$$

Problem 3.13 Colville 1 [3].

$$
\begin{aligned}
F(x) &= \sum_{j=1}^{5} d_j x_j^3 + \sum_{i=1}^{5}\sum_{j=1}^{5} c_{ij}x_i x_j + \sum_{j=1}^{5} e_j x_j + 50 \max\left[0, \max_{1\le i\le 10}\left(b_i - \sum_{j=1}^{5} a_{ij}x_j\right)\right],\\
A &= \begin{bmatrix} -16 & 2 & 0 & 1 & 0\\ 0 & -2 & 0 & 4 & 2\\ -3.5 & 0 & 2 & 0 & 0\\ 0 & -2 & 0 & -4 & -1\\ 0 & -9 & -2 & 1 & -2.8\\ 2 & 0 & -4 & 0 & 0\\ -1 & -1 & -1 & -1 & -1\\ -1 & -2 & -3 & -2 & -1\\ 1 & 2 & 3 & 4 & 5\\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} -40\\ -2\\ -0.25\\ -4\\ -4\\ -1\\ -40\\ -60\\ 5\\ 1 \end{bmatrix},\\
C &= \begin{bmatrix} 30 & -20 & -10 & 32 & -10\\ -20 & 39 & -6 & -31 & 32\\ -10 & -6 & 10 & -6 & -10\\ 32 & -31 & -6 & 39 & -20\\ -10 & 32 & -10 & -20 & 30 \end{bmatrix}, \qquad d = \begin{bmatrix} 4\\ 8\\ 10\\ 6\\ 2 \end{bmatrix},\\
e^T &= \begin{bmatrix} -15 & -27 & -36 & -18 & -12 \end{bmatrix},\\
\overline{x}_1 &= 0, \quad \overline{x}_2 = 0, \quad \overline{x}_3 = 0, \quad \overline{x}_4 = 0, \quad \overline{x}_5 = 1.
\end{aligned}
$$

Problem 3.14 HS78 [10].

$$
\begin{aligned}
F(x) &= x_1x_2x_3x_4x_5 + 10\sum_{i=1}^{3}|f_i(x)|,\\
f_1(x) &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - 10,\\
f_2(x) &= x_2x_3 - 5x_4x_5,\\
f_3(x) &= x_1^3 + x_2^3 + 1,\\
\overline{x}_1 &= -2.0, \quad \overline{x}_2 = 1.5, \quad \overline{x}_3 = 2.0,\\
\overline{x}_4 &= -1.0, \quad \overline{x}_5 = -1.0.
\end{aligned}
$$

Problem 3.15 El-Attar [7].

$$
\begin{aligned}
F(x) &= \sum_{i=1}^{51}\left|x_1 e^{-x_2 t_i}\cos(x_3 t_i + x_4) + x_5 e^{-x_6 t_i} - y_i\right|,\\
y_i &= 0.5e^{-t_i} - e^{-2t_i} + 0.5e^{-3t_i} + 1.5e^{-1.5t_i}\sin 7t_i + e^{-2.5t_i}\sin 5t_i,\\
t_i &= 0.1(i-1), \qquad 1 \le i \le 51,\\
\overline{x}_1 &= 2, \quad \overline{x}_2 = 2, \quad \overline{x}_3 = 7,\\
\overline{x}_4 &= 0, \quad \overline{x}_5 = -2, \quad \overline{x}_6 = 1.
\end{aligned}
$$

Problem 3.16 Maxquad [12] and [15].

$$\begin{aligned}
F(x) &= \max_{1\le i\le 5}\left(x^T A^i x - x^T b^i\right), \\
A^i_{kj} &= A^i_{jk} \;=\; e^{j/k}\cos(jk)\sin(i), \quad j<k, \\
A^i_{jj} &= \frac{j}{10}|\sin(i)| + \sum_{k\neq j}|A^i_{jk}|, \\
b^i_j &= e^{j/i}\sin(ij), \\
\overline{x}_i &= 0, \quad 1\le i\le 10.
\end{aligned}$$

Problem 3.17 Gill [11].

$$\begin{aligned}
F(x) &= \max\{f_1(x), f_2(x), f_3(x)\}, \\
f_1(x) &= \sum_{i=1}^{10}(x_i-1)^2 + 10^{-3}\sum_{i=1}^{10}\left(x_i^2-\frac{1}{4}\right)^2, \\
f_2(x) &= \sum_{i=2}^{30}\left[\sum_{j=2}^{10} x_j(j-1)\left(\frac{i-1}{29}\right)^{j-2} - \left(\sum_{j=1}^{10} x_j\left(\frac{i-1}{29}\right)^{j-1}\right)^2 - 1\right]^2 \\
&\quad + x_1^2 + (x_2 - x_1^2 - 1)^2, \\
f_3(x) &= \sum_{i=2}^{10}\left[100(x_i - x_{i-1}^2)^2 + (1-x_i)^2\right], \\
\overline{x}_i &= -0.1, \quad 1\le i\le 10.
\end{aligned}$$

Problem 3.18 Steiner 2 [11].

$$\begin{aligned}
F(x) &= \sqrt{x_1^2 + x_{1+m}^2} + \sqrt{(\overline{a}_{21}-x_m)^2 + (\overline{a}_{22}-x_{2m})^2} + \\
&\quad + \sum_{j=1}^{m} p_j\sqrt{(a_{j1}-x_j)^2 + (a_{j2}-x_{j+m})^2} + \\
&\quad + \sum_{j=1}^{m-1} \tilde{p}_j\sqrt{(x_j - x_{j+1})^2 + (x_{j+m}-x_{j+m+1})^2}, \qquad m=6,
\end{aligned}$$

$$\begin{array}{llll}
\overline{a}_{21} = 5.5, & \overline{a}_{22} = -1.0, & & \\
a_{11} = 0.0, & a_{12} = 2.0, & p_1 = 2, & \tilde{p}_1 = 1, \\
a_{21} = 2.0, & a_{22} = 3.0, & p_2 = 1, & \tilde{p}_2 = 1, \\
a_{31} = 3.0, & a_{32} = -1.0, & p_3 = 1, & \tilde{p}_3 = 2, \\
a_{41} = 4.0, & a_{42} = -0.5, & p_4 = 5, & \tilde{p}_4 = 3, \\
a_{51} = 5.0, & a_{52} = 2.0, & p_5 = 1, & \tilde{p}_5 = 2, \\
a_{61} = 6.0, & a_{62} = 2.0, & p_6 = 1, &
\end{array}$$

$$
\begin{aligned}
\overline{x}_1 &= (a_{11}+a_{21})/3, & \overline{x}_{1+m} &= (a_{12}+a_{22})/3, \\
\overline{x}_j &= \left(\overline{x}_{j-1}+a_{j1}+a_{(j+1)1}\right)/3, & \overline{x}_{j+m} &= \left(\overline{x}_{j-1+m}+a_{j2}+a_{(j+1)2}\right)/3, \quad 2 \le j \le m-1, \\
\overline{x}_m &= (\overline{x}_{m-1}+a_{m1}+\overline{a}_{21})/3, & \overline{x}_{2m} &= (\overline{x}_{2m-1}+a_{m2}+\overline{a}_{22})/3.
\end{aligned}
$$

Problem 3.19 Maxq [15].

$$
\begin{aligned}
F(x) &= \max_{1\le i\le 20} x_i^2, \\
\overline{x}_i &= i, \quad i=1,\dots,10, \qquad \overline{x}_i = -i, \quad i=11,\dots,20.
\end{aligned}
$$

Problem 3.20 Maxl [15].

$$
\begin{aligned}
F(x) &= \max_{1\le i\le 20} |x_i|, \\
\overline{x}_i &= i, \quad i=1,\dots,10, \qquad \overline{x}_i = -i, \quad i=11,\dots,20.
\end{aligned}
$$

Problem 3.21 TR48 [12] and [15].

$$
\begin{aligned}
F(x) &= \sum_{j=1}^{48} d_j \max_{1\le i\le 48}(x_i - a_{ij}) - \sum_{i=1}^{48} s_i x_i, \\
\overline{x}_i &= 0, \quad 1 \le i \le 48.
\end{aligned}
$$

Coefficients a_{ij}, s_i, d_j are given in [12].

Problem 3.22 Goffin [15].

$$
\begin{aligned}
F(x) &= 50 \max_{1\le i\le 50} x_i - \sum_{i=1}^{50} x_i, \\
\overline{x}_i &= i - 25.5, \quad 1 \le i \le 50.
\end{aligned}
$$

Problem 3.23 MXHILB [11].

$$
\begin{aligned}
F(x) &= \max_{1\le i\le 50} \left| \sum_{j=1}^{50} \frac{x_j}{i+j-1} \right|, \\
\overline{x}_i &= 1, \quad 1 \le i \le 50.
\end{aligned}
$$

Problem 3.24 L1HILB [11].

$$
\begin{aligned}
F(x) &= \sum_{i=1}^{50} \left| \sum_{j=1}^{50} \frac{x_j}{i+j-1} \right|, \\
\overline{x}_i &= 1, \quad 1 \le i \le 50.
\end{aligned}
$$

Problem 3.25 Shell Dual [8].

$$
\begin{aligned}
F(x) &= 2\left|\sum_{i=1}^{5} d_i x_{i+10}^3\right| + \sum_{i=1}^{5}\sum_{j=1}^{5} c_{ij} x_{i+10} x_{j+10} - \\
&\quad - \sum_{i=1}^{10} b_i x_i + 100 \left(\sum_{i=1}^{5} \max(0, P_i(x)) - Q(x) \right), \\
P_i(x) &= \sum_{j=1}^{10} a_{ji} x_j - 2\sum_{j=1}^{5} c_{ij} x_{j+10} - 3 d_i x_{i+10}^2 - e_i, \quad 1 \le i \le 5, \\
Q(x) &= \sum_{i=1}^{15} \min(0, x_i), \\
\overline{x}_i &= 10^{-4}, \quad 1 \le i \le 15, \quad i \ne 7, \qquad \overline{x}_7 \;=\; 60.
\end{aligned}
$$

Matrices A, b, C, d, e are the same as in Problem 3.13.

4 Test problems for linearly constrained minimax optimization

Calling statements have the form

```
CALL TILD22(N,NA,NB,NC,X,IX,XL,XU,IC,CL,CU,CG,FMIN,XMAX,NEXT,IERR),
CALL TAFU22(N,KA,X,FA,NEXT),
CALL TAGU22(N,KA,X,GA,NEXT),
CALL TAHD22(N,KA,X,hA,NEXT),
```

with the following significance:

- `TILD22` - determination of problem dimension `N, NA, NB, NC`, initiation of vector of variables `X`, definition of box constraints `IX, XL, XU`, definition of the general linear constraints `IC, CL, CU, CG`.
- `TAFU22` - evaluation of the `KA`-th partial function value `FA` at point `X`.
- `TAGU22` - evaluation of the `KA`-th partial function gradient `GA` at point `X`.
- `TAHD22` - evaluation of the `KA`-th partial function Hessian matrix `HA` at point `X`.

We seek a minimum of a minimax objective function

$$
F(x) = \max_{1 \le k \le n_A} (f_k(x)), \quad x \in R^n,
$$

with simple bounds

$$
\begin{aligned}
x_i - \text{unbounded} \quad &, \quad I_i^x = 0, \\
x_i^l \le x_i \quad &, \quad I_i^x = 1, \\
x_i \le x_i^u \quad &, \quad I_i^x = 2, \\
x_i^l \le x_i \le x_i^u \quad &, \quad I_i^x = 3, \\
x_i = x_i^l = x_i^u \quad &, \quad I_i^x = 5
\end{aligned}
$$

and general linear constraints

$$
\begin{aligned}
a_i^T x - \text{unbounded} \quad &, \quad I_i^c = 0, \\
c_i^l \le a_i^T x \quad &, \quad I_i^c = 1, \\
a_i^T x \le c_i^u \quad &, \quad I_i^c = 2, \\
c_i^l \le a_i^T x \le c_i^u \quad &, \quad I_i^c = 3, \\
a_i^T x = c_i^l = c_i^u \quad &, \quad I_i^c = 5
\end{aligned}
$$

from the starting point $\bar{x}$. The description of individual problems follows. Table 4.1 contains problem dimensions and optimum function values.

No.	Problem	n	n_A	n_C	Optimum value
4.1	MAD1	2	3	1	-0.38965952
4.2	MAD2	2	3	1	-0.33035714
4.3	MAD4	2	3	1	-0.44891079
4.4	MAD5	2	3	1	-0.42928061
4.5	PENTAGON	6	3	15	-1.85961870
4.6	MAD6	7	163	9	0.10183089
4.7	EQUIL	8	8	1	0
4.8	Wong 2	10	6	3	24.306209
4.9	Wong 3	20	14	4	133.72828
4.10	MAD8	20	38	0	0.50694799
4.11	BP filter	9	124	4	$0.27607734 \cdot 10^{-3}$
4.12	HS114	10	9	5	-1768.8070
4.13	Dembo 3	7	13	2	1227.2260
4.14	Dembo 5	8	4	3	7049.2480
4.15	Dembo 7	16	19	1	174.78699

Table 4.1

Problem 4.1 MAD1 [14].

$$
\begin{aligned}
F(x) &= \max_{1 \le i \le 3} f_i(x), \\
f_1(x) &= x_1^2 + x_2^2 + x_1 x_2 - 1, \\
f_2(x) &= \sin x_1,
\end{aligned}
$$

$$\begin{aligned} f_3(x) &= -\cos x_2, \\ c_1(x) &= x_1 + x_2 - 0.5 \ \geq \ 0 \\ \overline{x}_1 &= 1, \qquad \overline{x}_2 \ = \ 2. \end{aligned}$$

Problem 4.2 MAD2 [14].

$$\begin{aligned} F(x) &= \max_{1\leq i\leq 3} f_i(x), \\ f_i(x) &- \text{ as in Problem 4.1}, \\ c_1(x) &= -3x_1 - x_2 - 2.5 \ \geq \ 0, \\ \overline{x}_1 &= -2, \qquad \overline{x}_2 \ = \ -1. \end{aligned}$$

Problem 4.3 MAD4 [14].

$$\begin{aligned} F(x) &= \max_{1\leq i\leq 3} f_i(x), \\ f_1(x) &= -\exp(x_1 - x_2), \\ f_2(x) &= \sinh(x_1 - 1) - 1, \\ f_3(x) &= -\log(x_2) - 1, \\ c_1(x) &= 0.05x_1 - x_2 + 0.5 \ \geq \ 0, \\ \overline{x}_1 &= -1.00, \qquad \overline{x}_2 \ = \ 0.01. \end{aligned}$$

Problem 4.4 MAD5 [14].

$$\begin{aligned} F(x) &= \max_{1\leq i\leq 3} f_i(x), \\ f_i(x) &- \text{ as in Problem 4.3}, \\ c_1(x) &= -0.9x_1 + x_2 - 1 \ \geq \ 0, \\ \overline{x}_1 &= -1, \qquad \overline{x}_2 \ = \ 3. \end{aligned}$$

Problem 4.5 PENTAGON [19].

$$\begin{aligned} F(x) &= \max_{1\leq i\leq 3} f_i(x), \\ f_1(x) &= -\sqrt{(x_1 - x_3)^2 + (x_2 - x_4)^2}, \\ f_2(x) &= -\sqrt{(x_3 - x_5)^2 + (x_4 - x_6)^2}, \\ f_3(x) &= -\sqrt{(x_5 - x_1)^2 + (x_6 - x_2)^2}, \\ & x_i \cos\frac{2\pi j}{5} + x_{i+1}\sin\frac{2\pi j}{5} \leq 1, \\ i &= 1,3,5, \qquad j \ = \ 0,1,2,3,4. \end{aligned}$$

Problem 4.6 MAD6 [14].

$$
\begin{aligned}
F(x) &= \max_{1\le i\le 163} f_i(x),\\
f_i(x) &= \frac{1}{15} + \frac{2}{15}\sum_{j=1}^{7}\cos(2\pi x_j \sin\vartheta_i), \qquad 1\le i\le 163,\\
\vartheta_i &= \frac{\pi}{180}(8.5 + i0.5), \qquad 1\le i\le 163,\\
c_1(x) &= x_1 \ \ge\ 0.4,\\
c_2(x) &= -x_1 + x_2 \ \ge\ 0.4,\\
c_3(x) &= -x_2 + x_3 \ \ge\ 0.4,\\
c_4(x) &= -x_3 + x_4 \ \ge\ 0.4,\\
c_5(x) &= -x_4 + x_5 \ \ge\ 0.4,\\
c_6(x) &= -x_5 + x_6 \ \ge\ 0.4,\\
c_7(x) &= -x_6 + x_7 \ \ge\ 0.4,\\
c_8(x) &= -x_4 + x_6 \ =\ 1.0,\\
c_9(x) &= x_7 \ =\ 3.5,\\
\overline{x}_1 &= 0.5, \qquad \overline{x}_2 \ =\ 1.0, \qquad \overline{x}_3 \ =\ 1.5, \qquad \overline{x}_4 \ =\ 2.0,\\
\overline{x}_5 &= 2.5, \qquad \overline{x}_6 \ =\ 3.0, \qquad \overline{x}_7 \ =\ 3.5.
\end{aligned}
$$

Problem 4.7 EQUIL [12].

$$
\begin{aligned}
F(x) &= \max_{1\le i\le 8} f_i(x),\\
f_i(x) &= \sum_{j=1}^{5}\left(\frac{a_{ji}\sum_{k=1}^{8} w_{jk}x_k}{x_i^{b_j}\sum_{k=1}^{8} a_{jk}x_k^{1-b_j}} - w_{ji}\right),\\
c_1(x) &= \sum_{i=1}^{8} x_i = 1, \qquad x_i \ \ge\ 0, \qquad 1\le i\le 8,\\
W &= [w_{jk}] = \begin{bmatrix} 3 & 1 & 0.1 & 0.1 & 5 & 0.1 & 0.1 & 6\\ 0.1 & 10 & 0.1 & 0.1 & 5 & 0.1 & 0.1 & 0.1\\ 0.1 & 9 & 10 & 0.1 & 4 & 0.1 & 7 & 0.1\\ 0.1 & 0.1 & 0.1 & 10 & 0.1 & 3 & 0.1 & 0.1\\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 11 \end{bmatrix},\\
A &= [a_{jk}] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\ 2 & 0.8 & 1 & 0.5 & 1 & 1 & 1 & 1\\ 1 & 1.2 & 0.8 & 1.2 & 1.6 & 2 & 0.6 & 0.1\\ 2 & 0.1 & 0.6 & 2 & 1 & 1 & 1 & 2\\ 1.2 & 1.2 & 0.8 & 1 & 1.2 & 0.1 & 3 & 4 \end{bmatrix},\\
b &= [b_j] = \begin{bmatrix} 0.5 & 1.2 & 0.8 & 2.0 & 1.5 \end{bmatrix},
\end{aligned}
$$

$$\overline{x}_i = 0.125, \qquad 1 \le i \le 8.$$

Problem 4.8 Wong 2 [1].

$$\begin{aligned}
F(x) &= \max_{1\le i\le 6} f_i(x),\\
f_1(x) &= x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3-10)^2 + 4(x_4-5)^2 + (x_5-3)^2 +\\
&\quad +2(x_6-1)^2 + 5x_7^2 + 7(x_8-11)^2 + 2(x_9-10)^2 + (x_{10}-7)^2 + 45,\\
f_2(x) &= f_1(x) + 10(3(x_1-2)^2 + 4(x_2-3)^2 + 2x_3^2 - 7x_4 - 120),\\
f_3(x) &= f_1(x) + 10(5x_1^2 + 8x_2 + (x_3-6)^2 - 2x_4 - 40),\\
f_4(x) &= f_1(x) + 10(0.5(x_1-8)^2 + 2(x_2-4)^2 + 3x_5^2 - x_6 - 30),\\
f_5(x) &= f_1(x) + 10(x_1^2 + 2(x_2-2)^2 - 2x_1x_2 + 14x_5 - 6x_6),\\
f_6(x) &= f_1(x) + 10(-3x_1 + 6x_2 + 12(x_9-8)^2 - 7x_{10}),\\
c_1(x) &= 4x_1 + 5x_2 - 3x_7 + 9x_8 \le 105,\\
c_2(x) &= 10x_1 - 8x_2 - 17x_7 + 2x_8 \le 0,\\
c_3(x) &= -8x_1 + 2x_2 + 5x_9 - 2x_{10} \le 12,\\
\overline{x}_1 &= 2, \quad \overline{x}_2 = 3, \quad \overline{x}_3 = 5, \quad \overline{x}_4 = 5, \quad \overline{x}_5 = 1, \quad \overline{x}_6 = 2,\\
\overline{x}_7 &= 7, \quad \overline{x}_8 = 3, \quad \overline{x}_9 = 6, \quad \overline{x}_{10} = 10.
\end{aligned}$$

Problem 4.9 Wong 3 [1].

$$\begin{aligned}
F(x) &= \max_{1\le i\le 14} f_i(x),\\
f_1(x) &= x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3-10)^2 + 4(x_4-5)^2 + (x_5-3)^2 +\\
&\quad +2(x_6-1)^2 + 5x_7^2 + 7(x_8-11)^2 + 2(x_9-10)^2 + (x_{10}-7)^2 + (x_{11}-9)^2 +\\
&\quad +10(x_{12}-1)^2 + 5(x_{13}-7)^2 + 4(x_{14}-14)^2 + 27(x_{15}-1)^2 + x_{16}^4 + (x_{17}-2)^2 +\\
&\quad +13(x_{18}-2)^2 + (x_{19}-3)^2 + x_{20}^2 + 95,\\
f_2(x) &= f_1(x) + 10(3(x_1-2)^2 + 4(x_2-3)^2 + 2x_3^2 - 7x_4 - 120),\\
f_3(x) &= f_1(x) + 10(5x_1^2 + 8x_2 + (x_3-6)^2 - 2x_4 - 40),\\
f_4(x) &= f_1(x) + 10(0.5(x_1-8)^2 + 2(x_2-4)^2 + 3x_5^2 - x_6 - 30),\\
f_5(x) &= f_1(x) + 10(x_1^2 + 2(x_2-2)^2 - 2x_1x_2 + 14x_5 - 6x_6),\\
f_6(x) &= f_1(x) + 10(-3x_1 + 6x_2 + 12(x_9-8)^2 - 7x_{10}),\\
f_7(x) &= f_1(x) + 10(x_1^2 + 5x_{11} - 8x_{12} - 28),\\
f_8(x) &= f_1(x) + 10(4x_1 + 9x_2 + 5x_{13}^2 - 9x_{14} - 87),\\
f_9(x) &= f_1(x) + 10(3x_1 + 4x_2 + 3(x_{13}-6)^2 - 14x_{14} - 10),\\
f_{10}(x) &= f_1(x) + 10(14x_1^2 + 35x_{15} - 79x_{16} - 92),\\
f_{11}(x) &= f_1(x) + 10(15x_2^2 + 11x_{15} - 61x_{16} - 54),
\end{aligned}$$

$$
\begin{aligned}
f_{12}(x) &= f_1(x) + 10(5x_1^2 + 2x_2 + 9x_{17}^4 - x_{18} - 68),\\
f_{13}(x) &= f_1(x) + 10(x_1^2 - x_9 + 19x_{19} - 20x_{20} + 19),\\
f_{14}(x) &= f_1(x) + 10(7x_1^2 + 5x_2^2 + x_{19}^2 - 30x_{20}),\\
c_1(x) &= 4x_1 + 5x_2 - 3x_7 + 9x_8 \le 105,\\
c_2(x) &= 10x_1 - 8x_2 - 17x_7 + 2x_8 \le 0,\\
c_3(x) &= -8x_1 + 2x_2 + 5x_9 - 2x_{10} \le 12,\\
c_4(x) &= x_1 + x_2 + 4x_{11} - 21x_{12} \le 0,\\
\overline{x}_1 &= 2, \quad \overline{x}_2 = 3, \quad \overline{x}_3 = 5, \quad \overline{x}_4 = 5, \quad \overline{x}_5 = 1, \quad \overline{x}_6 = 2, \quad \overline{x}_7 = 7,\\
\overline{x}_8 &= 3, \quad \overline{x}_9 = 6, \quad \overline{x}_{10} = 10, \quad \overline{x}_{11} = 2, \quad \overline{x}_{12} = 2, \quad \overline{x}_{13} = 6, \quad \overline{x}_{14} = 15,\\
\overline{x}_{15} &= 1, \quad \overline{x}_{16} = 2, \quad \overline{x}_{17} = 1, \quad \overline{x}_{18} = 2, \quad \overline{x}_{19} = 1, \quad \overline{x}_{20} = 3.
\end{aligned}
$$

Problem 4.10 MAD8 [14].

$$
\begin{aligned}
F(x) &= \max_{1\le i\le 38} |f_i(x)|,\\
f_1(x) &= -1 + x_1^2 + \sum_{j=2}^{20} x_j,\\
f_i(x) &= -1 + c_i x_k^2 + \sum_{j=1, j\ne k}^{20} x_j, \quad 1 < i < 38,\\
f_{38}(x) &= -1 + x_{20}^2 + \sum_{j=1}^{19} x_j,\\
k &= (i+2)/2, \quad c_i = 1, \quad i = 2, 4, \ldots, 36,\\
k &= (i+1)/2, \quad c_i = 2, \quad i = 3, 5, \ldots, 37,\\
x_j &\ge 0.5, \quad 1 \le j \le 10,\\
\overline{x}_j &= 100, \quad 1 \le j \le 20.
\end{aligned}
$$

Problem 4.11 BP filter.

$$
\begin{aligned}
F(x) &= \max_{1\le i\le 124} f_i(x),\\
f_1(x) &= \varphi(x, t_0) - x_9 + a,\\
f_{2j}(x) &= \varphi(x, t_j) - x_9, \quad 1 \le j \le 61,\\
f_{2j+1}(x) &= x_9 - \varphi(x, t_j), \quad 1 \le j \le 61,\\
f_{124}(x) &= \varphi(x, t_{62}) - x_9 + a,\\
\varphi(x, t) &= \frac{1}{2}\sum_{i=1}^{4} \log\left((x_{i+4}^2 - t^2)^2 + t^2 x_i^2\right) - 4\log(x), \quad a = 3.0164,\\
t_0 &= 0.967320, \quad t_1 = 0.987423,\\
t_j &= t_1 + (j-1)(t_{61} - t_1), \quad 1 \le j \le 61,
\end{aligned}
$$

$$
\begin{aligned}
t_{61} &= 1.012577, \quad t_{62} = 1.032680, \\
c_i(x) &= x_{i+4} - 10000x_i \le 0, \quad 1 \le i \le 4, \\
\overline{x}_1 &= 0.398 \cdot 10^{-1}, \quad \overline{x}_2 = 0.968 \cdot 10^{-4}, \quad \overline{x}_3 = 0.103 \cdot 10^{-3}, \\
\overline{x}_4 &= 0.389 \cdot 10^{-1}, \quad \overline{x}_5 = 1.010, \quad \overline{x}_6 = 0.968, \quad \overline{x}_7 = 1.030, \\
\overline{x}_8 &= 0.981, \quad \overline{x}_9 = -11.6.
\end{aligned}
$$

Problem 4.12 HS114 [10]

$$
\begin{aligned}
F(x) &= \max_{1 \le i \le 9} f_i(x), \\
f_1(x) &= 5.04x_1 + 0.035x_2 + 10x_3 + 3.36x_5 - 0.063x_4x_7, \\
f_2(x) &= f_1(x) + 500\left(1.12x_1 + 0.13167x_1x_8 - 0.00667x_1x_8^2 - \frac{1}{a}x_4\right), \\
f_3(x) &= f_1(x) - 500\left(1.12x_1 + 0.13167x_1x_8 - 0.00667x_1x_8^2 - ax_4\right), \\
f_4(x) &= f_1(x) + 500\left(1.098x_8 - 0.038x_8^2 + 0.325x_6 - \frac{1}{a}x_7 + 57.425\right), \\
f_5(x) &= f_1(x) - 500\left(1.098x_8 - 0.038x_8^2 + 0.325x_6 - ax_7 + 57.425\right), \\
f_6(x) &= f_1(x) + 500\left(\frac{98000x_3}{x_4x_9 + 1000x_3} - x_6\right), \\
f_7(x) &= f_1(x) - 500\left(\frac{98000x_3}{x_4x_9 + 1000x_3} - x_6\right), \\
f_8(x) &= f_1(x) + 500\left(\frac{x_2 + x_5}{x_1} - x_8\right), \\
f_9(x) &= f_1(x) - 500\left(\frac{x_2 + x_5}{x_1} - x_8\right), \\
c_1(x) &= 0.222x_{10} + bx_9 \le 35.82, \\
c_2(x) &= 0.222x_{10} + \frac{1}{b}x_9 \ge 35.82, \\
c_3(x) &= 3x_7 - ax_{10} \ge 133, \\
c_4(x) &= 3x_7 - \frac{1}{a}x_{10} \le 133, \\
c_5(x) &= 1.22x_4 - x_1 - x_5 = 0, \\
a &= 0.99, \quad b = 0.90,
\end{aligned}
$$

$$
\begin{array}{rcccll}
10^{-5} & \le & x_1 & \le & 2000, & \overline{x}_1 = 1745, \\
10^{-5} & \le & x_2 & \le & 16000, & \overline{x}_2 = 12000, \\
10^{-5} & \le & x_3 & \le & 120, & \overline{x}_3 = 110, \\
10^{-5} & \le & x_4 & \le & 5000, & \overline{x}_4 = 3048, \\
10^{-5} & \le & x_5 & \le & 2000, & \overline{x}_5 = 1974, \\
85 & \le & x_6 & \le & 93, & \overline{x}_6 = 89.2, \\
90 & \le & x_7 & \le & 95, & \overline{x}_7 = 92.8, \\
3 & \le & x_8 & \le & 12, & \overline{x}_8 = 8.0, \\
1.2 & \le & x_9 & \le & 4, & \overline{x}_9 = 3.6, \\
145 & \le & x_{10} & \le & 162, & \overline{x}_{10} = 145.
\end{array}
$$

Problem 4.13 Dembo 3 [6].

$$
\begin{array}{rcl}
F(x) & = & \max_{1\le i\le 13} f_i(x), \\
f_1(x) & = & a_1x_1 + a_2x_1x_6 + a_3x_3 + a_4x_2 + a_5 + a_6x_3x_5, \\
f_2(x) & = & f_1(x) + 10^5\left(a_7x_6^2 + a_8x_1^{-1}x_3 + a_9x_6 - 1\right), \\
f_3(x) & = & f_1(x) + 10^5\left(a_{10}x_1x_3^{-1} + a_{11}x_1x_3^{-1}x_6 + a_{12}x_1x_3^{-1}x_6^2 - 1\right), \\
f_4(x) & = & f_1(x) + 10^5\left(a_{13}x_6^2 + a_{14}x_5 + a_{15}x_4 + a_{16}x_6 - 1\right), \\
f_5(x) & = & f_1(x) + 10^5\left(a_{17}x_5^{-1} + a_{18}x_5^{-1}x_6 + a_{19}x_4x_5^{-1} + a_{20}x_5^{-1}x_6^2 - 1\right), \\
f_6(x) & = & f_1(x) + 10^5\left(a_{21}x_7 + a_{22}x_2x_3^{-1}x_4^{-1} + a_{23}x_2x_3^{-1} - 1\right), \\
f_7(x) & = & f_1(x) + 10^5\left(a_{24}x_7^{-1} + a_{25}x_2x_3^{-1}x_7^{-1} + a_{26}x_2x_3^{-1}x_4^{-1}x_7^{-1} - 1\right), \\
f_8(x) & = & f_1(x) + 10^5\left(a_{27}x_5^{-1} + a_{28}x_5^{-1}x_7 - 1\right), \\
f_9(x) & = & f_1(x) + 10^5\left(a_{33}x_1x_3^{-1} + a_{34}x_3^{-1} - 1\right), \\
f_{10}(x) & = & f_1(x) + 10^5\left(a_{35}x_2x_3^{-1}x_4^{-1} + a_{36}x_2x_3^{-1} - 1\right), \\
f_{11}(x) & = & f_1(x) + 10^5\left(a_{37}x_4 + a_{38}x_2^{-1}x_3x_4 - 1\right), \\
f_{12}(x) & = & f_1(x) + 10^5\left(a_{39}x_1x_6 + a_{40}x_1 + a_{41}x_3 - 1\right), \\
f_{13}(x) & = & f_1(x) + 10^5\left(a_{42}x_1^{-1}x_3 + a_{43}x_1^{-1} + a_{44}x_6 - 1\right), \\
c_1(x) & = & a_{29}x_5 + a_{30}x_7 \le 1, \\
c_2(x) & = & a_{31}x_3 + a_{32}x_1 \le 1,
\end{array}
$$

i	a_i	i	a_i	i	a_i
1	1.715	16	$-0.19120592 \cdot 10^{-1}$	31	0.00061000
2	0.035	17	$0.56850750 \cdot 10^{2}$	32	-0.0005
3	4.0565	18	1.08702000	33	0.81967200
4	10.0	19	0.32175000	34	0.81967200
5	3000.0	20	-0.03762000	35	24500.0
6	-0.063	21	0.00619800	36	-250.0
7	$0.59553571 \cdot 10^{-2}$	22	$0.24623121 \cdot 10^{4}$	37	$0.10204082 \cdot 10^{-1}$
8	0.88392857	23	$-0.25125634 \cdot 10^{2}$	38	$0.12244898 \cdot 10^{-4}$
9	-0.11756250	24	$0.16118996 \cdot 10^{3}$	39	0.00006250
10	1.10880000	25	5000.0	40	0.00006250
11	0.13035330	26	$-0.48951000 \cdot 10^{6}$	41	-0.00007625
12	-0.00660330	27	$0.44333333 \cdot 10^{2}$	42	1.22
13	$0.66173269 \cdot 10^{-3}$	28	0.33000000	43	1.0
14	$0.17239878 \cdot 10^{-1}$	29	0.02255600	44	-1.0
15	$-0.56595559 \cdot 10^{-2}$	30	-0.00759500		

$$
\begin{array}{rcccrl}
1 & \leq & x_1 & \leq & 2000 & , \quad \overline{x}_1 = 1745, \\
1 & \leq & x_2 & \leq & 120 & , \quad \overline{x}_2 = 110, \\
1 & \leq & x_3 & \leq & 5000 & , \quad \overline{x}_3 = 3048, \\
85 & \leq & x_4 & \leq & 93 & , \quad \overline{x}_4 = 89, \\
90 & \leq & x_5 & \leq & 95 & , \quad \overline{x}_3 = 92, \\
3 & \leq & x_6 & \leq & 12 & , \quad \overline{x}_4 = 8, \\
145 & \leq & x_7 & \leq & 162 & , \quad \overline{x}_5 = 145.
\end{array}
$$

Problem 4.14 Dembo 5 [6].

$$
\begin{array}{rcl}
F(x) & = & \max\limits_{1 \leq i \leq 4} f_i(x), \\
f_1(x) & = & x_1 + x_2 + x_3, \\
f_2(x) & = & f_1(x) + 10^5 \left(a x_1^{-1} x_4 x_6^{-1} + 100 x_6^{-1} + b x_1^{-1} x_6^{-1} - 1 \right), \\
f_3(x) & = & f_1(x) + 10^5 \left(x_4 x_7^{-1} + 1250(x_5 - x_4) x_2^{-1} x_7^{-1} - 1 \right), \\
f_4(x) & = & f_1(x) + 10^5 \left(c x_3^{-1} x_8^{-1} + x_5 x_8^{-1} - 2500 x_3^{-1} x_5 x_8^{-1} - 1 \right), \\
c_1(x) & = & 0.0025(x_4 + x_6) \leq 1, \\
c_2(x) & = & 0.0025(x_5 + x_7 - x_4) \leq 1, \\
c_3(x) & = & 0.01(x_8 - x_5) \leq 1, \\
a & = & 833.33252, \quad b = -83333.333, \quad c = 1250000.0.
\end{array}
$$

$$
\begin{array}{rcrcl}
100 & \leq & x_1 & \leq & 10000 \quad , \quad \overline{x}_1 = 5000, \\
1000 & \leq & x_2 & \leq & 10000 \quad , \quad \overline{x}_2 = 5000, \\
1000 & \leq & x_3 & \leq & 10000 \quad , \quad \overline{x}_3 = 5000, \\
10 & \leq & x_4 & \leq & 1000 \quad , \quad \overline{x}_4 = 200, \\
10 & \leq & x_5 & \leq & 1000 \quad , \quad \overline{x}_5 = 350, \\
10 & \leq & x_6 & \leq & 1000 \quad , \quad \overline{x}_6 = 150, \\
10 & \leq & x_7 & \leq & 1000 \quad , \quad \overline{x}_7 = 225, \\
10 & \leq & x_8 & \leq & 1000 \quad , \quad \overline{x}_8 = 425.
\end{array}
$$

Problem 4.15 Dembo 7 [6].

$$
\begin{array}{rcl}
F(x) & = & \max\limits_{1 \leq i \leq 19} f_i(x), \\
f_1(x) & = & a(x_{12} + x_{13} + x_{14} + x_{15} + x_{16}) + b(x_1 x_{12} + x_2 x_{13} + x_3 x_{14} + x_4 x_{15} + x_5 x_{16}), \\
f_2(x) & = & f_1(x) + 10^3 \left(c x_1 x_6^{-1} + 100 d x_1 - d x_1^2 x_6^{-1} - 1 \right), \\
f_3(x) & = & f_1(x) + 10^3 \left(c x_2 x_7^{-1} + 100 d x_2 - d x_2^2 x_7^{-1} - 1 \right), \\
f_4(x) & = & f_1(x) + 10^3 \left(c x_3 x_8^{-1} + 100 d x_3 - d x_3^2 x_8^{-1} - 1 \right), \\
f_5(x) & = & f_1(x) + 10^3 \left(c x_4 x_9^{-1} + 100 d x_4 - d x_4^2 x_9^{-1} - 1 \right), \\
f_6(x) & = & f_1(x) + 10^3 \left(c x_5 x_{10}^{-1} + 100 d x_5 - d x_5^2 x_{10}^{-1} - 1 \right), \\
f_7(x) & = & f_1(x) + 10^3 \left(x_6 x_7^{-1} + x_1 x_7^{-1} x_{11}^{-1} x_{12} - x_6 x_7^{-1} x_{11}^{-1} x_{12} - 1 \right), \\
f_8(x) & = & f_1(x) + 10^3 \left(x_7 x_8^{-1} + 0.002(x_7 - x_1) x_8^{-1} x_{12} + 0.002(x_2 x_8^{-1} - 1) x_{13} - 1 \right), \\
f_9(x) & = & f_1(x) + 10^3 \left(x_8 + 0.002(x_8 - x_2) x_{13} + 0.002(x_3 - x_9) x_{14} + x_9 - 1 \right), \\
f_{10}(x) & = & f_1(x) + 10^3 \left(x_3^{-1} x_9 + (x_4 - x_8) x_3^{-1} x_{14}^{-1} x_{15} + 500(x_{10} - x_9) x_3^{-1} x_{14}^{-1} - 1 \right), \\
f_{11}(x) & = & f_1(x) + 10^3 \left((x_4^{-1} x_5 - 1) x_{15}^{-1} x_{16} + x_4^{-1} x_{10} + 500(1 - x_4^{-1} x_{10}) x_{15}^{-1} - 1 \right), \\
f_{12}(x) & = & f_1(x) + 10^3 \left(0.9 x_4^{-1} + 0.002(1 - x_4^{-1} x_5) x_{16} - 1 \right), \\
f_{13}(x) & = & f_1(x) + 10^3 \left(x_{11}^{-1} x_{12} - 1 \right), \\
f_{14}(x) & = & f_1(x) + 10^3 \left(x_4 x_5^{-1} - 1 \right), \\
f_{15}(x) & = & f_1(x) + 10^3 \left(x_3 x_4^{-1} - 1 \right), \\
f_{16}(x) & = & f_1(x) + 10^3 \left(x_2 x_3^{-1} - 1 \right), \\
f_{17}(x) & = & f_1(x) + 10^3 \left(x_1 x_2^{-1} - 1 \right), \\
f_{18}(x) & = & f_1(x) + 10^3 \left(x_9 x_{10}^{-1} - 1 \right), \\
f_{19}(x) & = & f_1(x) + 10^3 \left(x_8 x_9^{-1} - 1 \right), \\
c_1(x) & = & 0.002(x_{11} - x_{12}) \leq 1, \\
a & = & 1.262626, \quad b = -1.231060, \quad c = 0.034750, \quad d = 0.009750.
\end{array}
$$

$$
\begin{array}{rcccccl}
0.1 & \leq & x_1 & \leq & 0.9 & , & \overline{x}_1 = 0.80, \\
0.1 & \leq & x_2 & \leq & 0.9 & , & \overline{x}_2 = 0.83, \\
0.1 & \leq & x_3 & \leq & 0.9 & , & \overline{x}_3 = 0.85, \\
0.1 & \leq & x_4 & \leq & 0.9 & , & \overline{x}_4 = 0.87, \\
0.9 & \leq & x_5 & \leq & 1.0 & , & \overline{x}_5 = 0.90, \\
10^{-4} & \leq & x_6 & \leq & 0.1 & , & \overline{x}_6 = 0.10, \\
0.1 & \leq & x_7 & \leq & 0.9 & , & \overline{x}_7 = 0.12, \\
0.1 & \leq & x_8 & \leq & 0.9 & , & \overline{x}_8 = 0.19, \\
0.1 & \leq & x_9 & \leq & 0.9 & , & \overline{x}_9 = 0.25, \\
0.1 & \leq & x_{10} & \leq & 0.9 & , & \overline{x}_{10} = 0.29, \\
1 & \leq & x_{11} & \leq & 1000 & , & \overline{x}_{11} = 512, \\
10^{-6} & \leq & x_{12} & \leq & 500 & , & \overline{x}_{12} = 13.1, \\
1 & \leq & x_{13} & \leq & 500 & , & \overline{x}_{13} = 71.8, \\
500 & \leq & x_{14} & \leq & 1000 & , & \overline{x}_{14} = 640, \\
500 & \leq & x_{15} & \leq & 1000 & , & \overline{x}_{15} = 650, \\
10^{-6} & \leq & x_{16} & \leq & 500 & , & \overline{x}_{16} = 5.7.
\end{array}
$$

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