APPLICATIONS OF TRANSFORMATION THEORY: A LEGACY FROM ZOLOTAREV (1847-1878)*

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"And out of olde bokes, in good feyth,
Cometh all this newe science that men lere"

CHAUCER, THE PARLIAMENT OF FOWLS

INTRODUCTION

What we are concerned with is, roughly, the generalization to the elliptic case of the familiar multiple angle formulas of elementary trigonometry such as

$$\cos 2\theta = 2\cos^2\theta - 1; \quad \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta};$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{(1-\sin^2\theta)}$$

(which are respectively polynomial, rational, algebraic). More generally we have

$$\cos n\theta = 2^{n-1}[\cos^n\theta - \tfrac{1}{4}n\cos^{n-2}\theta + \ldots]$$

which we can also express as a Chebyshev polynomial:

$$T_n(x) = \cos(n \arccos x) = 2^{n-1}[x^n - \tfrac{1}{4}nx^{n-2} + \ldots]$$

$$= 2^{n-1}\tilde{T}_n(x).$$

S. P. Singh et al. (eds.), Approximation Theory and Spline Functions, 207–245.

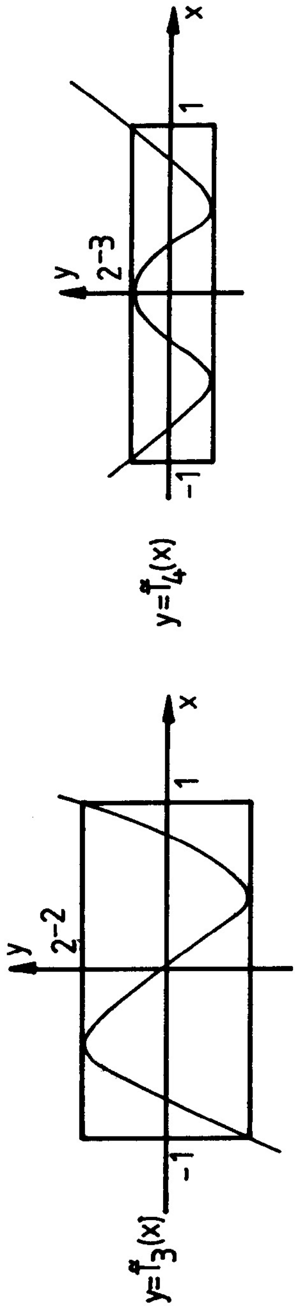


Figure 1

Zolotarev stated and solved in 1868, 1877, 1878, four problems in approximation theory, or the constructive theory of functions. These problems turned up again in practical contexts in different areas and different countries and were solved independently during the last 50 years. We state these problems in §2. We describe in 3A, B, C some applications of (Z3). Finally, we give some indication of the solutions: in §4, we first discuss an elementary problem which indicates the method for dealing with (Z1) and in §5, we discuss the general method of solution. The solutions all depend on the theory of transformation of elliptic functions, a subject beyond the scope of the usual texts, [cf. 22.421, W8W] and the usual sylllabi. Greenhill [1892, p. x, Introduction] notes the reintroduction of elliptic functions "... excluding the theta functions and the theory of transformation" in the regulations for Schedule II, Part I of the Mathematical Tripos at Cambridge, beginning in May 1893.

Although Chebyshev was well aware of the inspiration afforded by applications, as indicated by the following quotation [Chebyshev, 1899, I, p. 239] there seems to be no reference to the potentialities of the work of Zolotarev.

"Le rapprochement de la théorie et de la pratique donnent les résultats les plus féconds. La pratique n'est pas la seule à tirer profit de ces rapports: réciproquement les sciences elles-mêmes se développent sous l'influence de la pratique. C'est elle qui leur découvre de nouveaux sujets d'étude et des points de vue nouveaux sur les sujets connus depuis longtemps."

There is a short biography of Zolotarev by Ozigova [1966]. Actually he is perhaps more celebrated for his work in algebra and number theory than in approximation theory.

For an account of Chebyshev's visit to England in 1852 and other relevant matters, see the Inaugural Lecture of A. Talbot [1971].

It is worth noting that Zolotarev wrote in the Minutes of the Meeting of the Council of the St. Petersburg University for the second half of the academic year 1869/70: "In mathematics it is incomparably harder to find a problem and state in correctly than to solve it; as soon as a problem is stated correctly its solution is found in one way or another."

See Kuznetsov [1971].

§2. ZOLOTAREV'S FOUR PROBLEMS

To see the place of the first problem we go back to the Chebyshev polynomials. It is well known that the best approximation to zero in $[-1,1]$ by a monic polynomial in the Chebyshev norm is $\tilde{T}_n(x)$. In fact

(T1) $$\min_{(a)} \max_{-1\le x\le 1} |x^n + a_1 x^{n-1} + \dots + a_n|$$

<u>is</u> 2^{1-n} <u>and is achieved by</u>

$$\tilde{T}_n(x) = \Pi(x - x_r)$$

<u>where</u>

$$x_r = \cos((2r+1)\pi/2n),\ r = 0, 1, \dots, n-1.$$

There are several related problems which we state:

(T2) (Markov) <u>Determine</u>

$$\min_{(a)} \max_{-1\le x\le 1} |a_0 x^n + a_1 x^{n-1} + \dots + a_n|$$

<u>where</u> $a_r = 1$ <u>for some</u> r, $1 \le r \le n$.

(T3) (Chebyshev) <u>Determine</u>

$$\min_{(a)} \max_{-1\le x\le 1} |a_0 x^n + a_1 x^{n-1} + \dots + a_n|$$

<u>where for some</u> ξ <u>outside</u> $[-1,1]$, $a_0\xi^n + a_1\xi^{n-1} + \dots + a_n = \eta$, η <u>given</u>.

We note that the T_n's, in compensation for their smallness inside $[-1,1]$, are largest outside:

(T4) (Chebyshev) <u>If</u> $p_n(x)$ <u>is a polynomial of degree</u> n <u>such that</u> $\max_{-1\le x\le 1} |p_n(x)| = 1$ <u>then for</u> ξ <u>outside</u> $[-1,1]$ <u>we have</u>

$$|p_n(\xi)| \le |T_n(\xi)|.$$

In (T1) Chebyshev fixed the *first* coefficent. Zolotarev asked the same question only requiring that the first *two* coefficients be fixed.

(Z1) Determine

$$\min_{(a)} \max_{-1\le x\le 1} |x^n - n\sigma x^{n-1} + a_2 x^{n-2} + \ldots + a_n|$$

where σ is a parameter.

This being solved, it is natural to ask the same question only fixing the first *three* coefficients. This was solved by Achiezer in 1928. The final stage was results about the case when r coefficients were fixed: these were obtained by Meiman in 1960. For details see the reviews and translations of his papers.

The second problem of Zolotarev is related to (Z1) just as (T3) is to (T1).

(Z2) Determine

$$\min_{(a_2,\ldots,a_n)} \max_{-1\le x\le 1} |x^n + a_1 x^{n-1} + \ldots + a_n|$$

where a_1 is determined so that $\xi^n + a_1\xi^{n-1} + \ldots + a_n = \eta$, where ξ, ($\xi < -1$ or $\xi > 1$) and η are given.

The other two problems of Zolotarev are concerned with *rational* approximation. [Compare these with (T4).] The relations between the problems have been discussed in detail by Achiezer.

(Z3) Find the rational function $y = \phi(x)/\psi(x)$, where the degrees of the polynomials ϕ, ψ do not exceed n, which satisfies

$$|y(x)| \le 1, \ -1 \le x \le 1$$

and which deviates most from zero in the intervals $|x| \ge k^{-1}$, where k, $0 < k < 1$ is given.

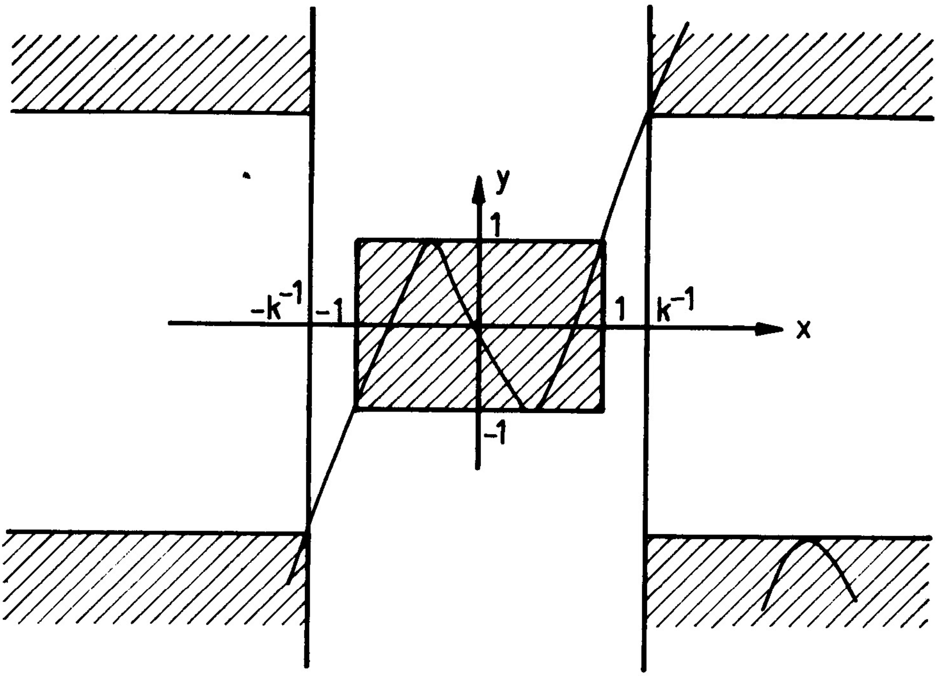


Figure 2

(Z4) Find the rational function $y = \phi(x)/\psi(x)$, where the degrees of the polynomials ϕ, ψ do not exceed n, such that

$$y(x) \geq 1 \quad \text{for} \quad 1 \leq x \leq k^{-1}, \; y(x) \leq -1 \quad \text{for} \quad -k^{-1} \leq x \leq -1$$

and which deviates least from zero in these intervals, where $0 < k < 1$.

We mention here a problem discussed by Achiezer:

(A1) Determine

$$\min_{(a)} \max |x^n + a_1 x^{n-1} + \dots + a_n|$$

where the max is over all x in the *two* intervals $-1 \leq x \leq -\lambda$, $\lambda \leq x \leq 1$ where λ, $0 < \lambda < 1$ is given.

§3. SOME APPLICATIONS OF ZOLOTAREV'S THIRD PROBLEM (Z3)

A. Design of Filters

An electrical filter is a "black box" with "knobs", containing variable components (condensers, resistances), which influences an input signal according to a response curve:

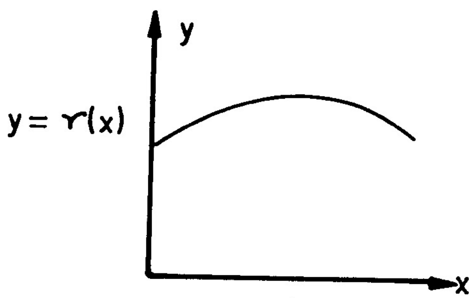


Figure 3

We have contact every day with filters: implicitly in telephone conversations and explictly in high-fidelity equipment. For a more detailed discussion see Melzak [1976].

To filter out the "high notes" requires a response of the form:

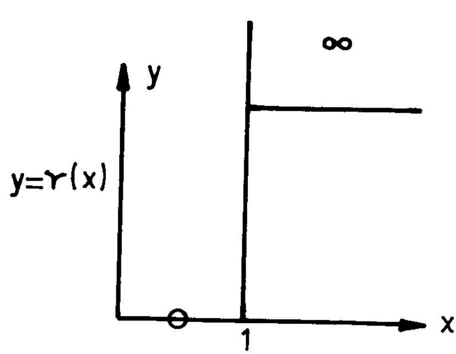


Figure 4

This is a "low pass" filter. This realises the truncation of a

Fourier series. This steep cut-off is not realisable and so we have to modify our demand to

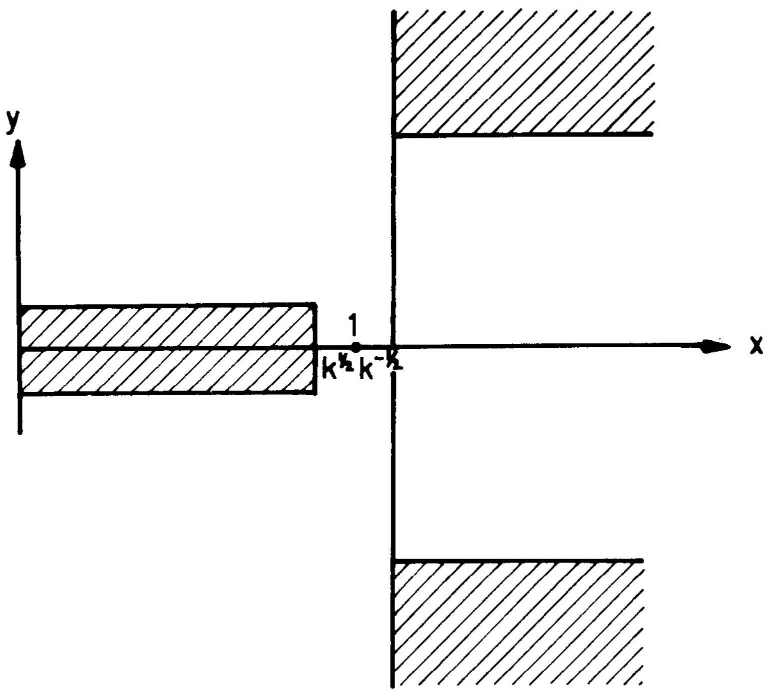


Figure 5

i.e. we want to adjust the parameters so that, given k, $0 < k < 1$

$|r|$ is small in $[0,k^{1/2}]$, $|r|$ is large in $[k^{-1/2},\infty)$.

For certain circuits the response is of the form

$$r(x) = \Pi|(a_j^2 - x^2)/(1 - a_j^2x^2)|$$

and so our problem becomes

(Z3') <u>Determine</u>

$$\min_{(a)} \max_{0\le x\le\sqrt{k}} |\Pi(a_j^2 - x^2)/(1-a_j^2x^2)|$$

which is very similar to (T1) when written as

(T1') <u>Determine</u> $\min_{(r)} \max_{-1\le x\le 1} \Pi(x - r_j)$.

[Because $r(x)\, r(x^{-1}) = 1$ we have only to consider one of the conditions (1).]

These problems were discussed in Germany, beginning with W. Cauer [1933] and in U.S.A. at Bell Telephone Laboratories [1939] by S. D. Darlington and E. L. Norton. Cauer was employed at the Mix and Genest organisation, later a subsidiary of the ITT Corporation.

The solution to (Z3') is given by

$$a_j = \sqrt{k}\ \mathrm{sn}(2jK/(2m+1),k) \quad j = 1, 2, \ldots, m$$

and the extremal value is

$$\sqrt{k_{2m}} = \sqrt{\frac{1-k'_m}{1+k'_m}}$$

where k_m corresponds to q^m as k corresponds to q.

The number n determines the size (cost) of the filter, the parameter k determines how sharp the cut-off is and the min-max gives the attenuation in the pass-band. To assist the designer, tables of the quantities involved were made -- now-a-days program packages would be written -- by e.g., Glowatzki [1955] U.S. National Bureau of Standards, (1956, unpublished), and A. R. Curtis [1964].

E. Stiefel [1961] contemplated extensions of this problem where we want to filter out a band (or several bands) of frequencies

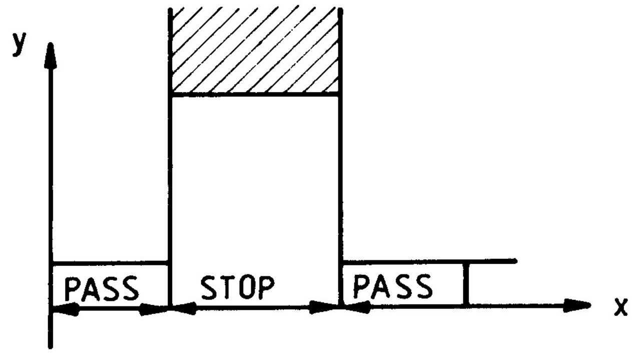


Figure 6

The solution to this problem involves hyperelliptic functions. Stiefel's associates, Amer and Schwarz [1964] have solved some practical problems, not in closed form, but by linear programming methods. This work was subsidized by the Hasler foundation.

We mention here that a problem concerning the optimal design of radio transmitters which produce a narrow principal beam and small subsidiary beams has been discussed by Pokrovskii [1962]. This leads to an extremal problem which generalizes (A1).

3B. ADI Parameters

The alternating direction implicit method for the iterative solution of the discrete approximations to elliptic partial differential equations was introduced in 1955 by Peaceman and Rachford [1955] in work supported by Humble Oil Company. In the case of the "model problem" the speed of convergence depends on

$$\max_{k\le x\le 1}\left|\prod_{j=1}^{m}\{(x-r_j)/(x+r_j)\}\right|$$

where the r_j are certain parameters to be chosen and where k, l are lower and upper bounds to the characteristic values of the (normalized) matrix of the system of linear equations approximating the differential equation $u_{xx} + u_{yy} = 0$ in a square $0 \le x, y \le 1$. The question of the optimal choice of the r_j

was answered when $m = 2^{\ell}$ by Gastinel [1962] and Wachspress [1963]. Indeed, more generally, from an optimal set of m parameters an optimal set of $2m$ can be found by the use of the arithmetic-geometric mean. The optimal parameters were found by W. B. Jordan (see Wachspress [1963]), based on Cauer's work. Jordan and Wachspress were employed at the Knolls Atomic Power Laboratory of the General Electric Company.

The actual result is the following:

$$r_j = \mathrm{dn}((2j-1)K/2m,k), \quad j = 1, 2, \ldots, m$$

$$L = L_m = (1 - \sqrt{k'_m})/(1 + \sqrt{k'_m})$$

where k_m corresponds to q^m as k corresponds to q.

The question of the optimal value of m arises. The proper question is about the behavior of

$$\eta_m = L_m^{1/m} .$$

Gaier and Todd [1967] showed that $\eta_m \uparrow$ and in fact that

$$\log \eta_m = \log q + [(\log 2)/m] + O(m^{-2}) \tag{2}$$

which implies that the asymptotic value was attained (in the practical range of q) for moderate values of m: favourite values of m were 8 or 16 or 10. De Boor and Rice [1963] had obtained empirical results which were very close to (2).

Another question which was recently studied by V. I. Lebedev [1977], was: What is the best order to use the parameters?

3C. Square Roots

Consider the determination of $\sqrt{N}$ by the Newton process: "guess x_0 and improve by $x_{n+1} = \frac{1}{2}(x_n + (N/x_n))$". Convergence takes place for any $x_0 > 0$. What is the best x_0 to use? We have to make this question more precise. Let us only consider using floating point calculators. It is then natural to restrict N to lie between 10^{-2} and 1 (in the decimal case) or between $\frac{1}{4}$ and 1 (in the binary case). It is then appropriate to consider the relative error

$$r_n = |(x_n - \sqrt{N})/\sqrt{N}|.$$

Next we ask: What value of n? In virtue of a minor miracle, first observed by Moursund [1967], it does not matter, provided $n \geq 1$. It is natural to consider making x_0 a rational function of N of type (μ,ν), say, i.e. numerator with degree $\leq \mu$, denominator with degree $\leq \nu$,

$$x_0 = n_\mu(N)/d_\nu(N)$$

so that we consider

$$r_1 = |1 - (x_1/\sqrt{N})|.$$

In virtue of another minor miracle the extremal x_1 is a constant multiple of the extremal x_0 so we have to look at

$$|1 - \{n_\mu(N)/\sqrt{N}\, d_\nu(N)\}| .$$

For a collection of references to work in this area, see Todd [1977].

The problem of determining the optimal coefficients in n_μ, d_ν when $\mu = \nu$ or $\mu = \nu + 1$ was solved in general by Ninomiya [1970]. (For small μ, ν the coefficients were found algebraically by Maehly (see Cody [1964]).

A typical numerical result in the case of $(2,1)$ approximation in $(\frac{1}{4}, 1)$ is

$$x_0 = \frac{0.3432201292\ N^2 + 1.071299971\ N + 0.085805032}{N + 0.5}$$

for which we have

$N = 0.25$	$x_0 = 0.50020\ 0044$	$N = 1$	$x_0 = 1.00090\ 088$
	$x_1 = 0.50000\ 0040$		$x_1 = 1.00000\ 0451$

The general solution in the (n,n) case is

$$x_0 = \frac{2}{1+\lambda'} \quad \Pi \frac{(1 - c_{2r}) + c_{2r}x}{(1 - c_{2r-1}) + c_{2r-1}x} \tag{3}$$

where

$$c_r = \mathrm{sn}^2(rK/2n,k), \qquad \lambda = k^{2n} \, \Pi c_{2r-1}^2$$

and the min-max is

$$L = (1-\lambda')/(1+\lambda').$$

We shall derive the general formula in §5 below and obtain this one by specialization.

In a memorandum of 25 June, 1962, E. L. Wachspress used the A.G.M. parameters to accelerate the convergence of the Newton process for the positive square root of a positive definite matrix.

§4. ZOLOTAREV'S FIRST PROBLEM (Z1)

The applications of this seem less interesting. For instance, we can "economize" polynomials approximating a polynomial of degree n by one of degree $n - 2$. These ideas have been exploited by C. Lanczos [1893-1974 and S. Paszkowski [1962].

However the solution to the problem looks quite mysterious.

$$\text{(Z1)} \quad \min_{(a)} \ \max_{-1\le x\le 1} \left| x^n - n\sigma x^{n-1} + a_2 x^{n-2} + \dots + a_n \right| .$$

We may assume $\sigma \geq 0$. For $\sigma = 0$ we are back to (T1).

For small σ the solution is a distorted Chebyshev polynomial. Specifically, if $0 \leq \sigma \leq \tan^2(\pi/2n)$, the extremal polynomial is

$$2^{1-n}(1+\sigma)^n \, T_n((x-\sigma)/(\sigma+1)),$$

and the minmax is $2^{1-n}(1+\sigma)^n$.

For larger σ the solution is given in an extremely complicated form, involving various elliptic quantities. We use the standard notation of Whittaker and Watson [1927]. We are given n, σ and first solve the following equation for k, which is involved in K and in the elliptic and theta functions

$$* \quad 1+\sigma = \frac{2\ \mathrm{sn}\,(K/n)}{\mathrm{cn}(K/n)\,\mathrm{dn}(K/n)}\ \{\mathrm{ns}(2K/n) - \frac{\tilde{I}_4'(\pi/2n)}{\tilde{I}_4(\pi/2n)}\}\ .$$

This k is unique and $0 < k < 1$ when $\sigma > \tan^2(\pi/2n)$.

Then the extremal polynomial $y = x^n - n\sigma x^{n-1} + \ldots$ is given parametrically by

$$y = \frac{1}{2} L[X^n + X^{-n}],$$

$$x = [\mathrm{sn}^2 u + \mathrm{sn}^2(K/n)]/[\mathrm{sn}^2 u - \mathrm{sn}^2(K/n)],$$

where

$$X = [\tilde{I}_1((\pi/2n) - (\pi u/2K))]/[\tilde{I}_1((\pi/2n) + (\pi u/2K))].$$

When u runs from 0 to iK' then x runs from -1 to 1. The corresponding min max is

$$L = 2^{1-n}[k^{1/2}\tilde{I}_3^2/\{\tilde{I}_2(\pi/2n)\tilde{I}_3(\pi/2n)\}]^{2n}.$$

A similar separation into cases occurs in the solution of (A1). Here the solution is a distorted Chebyshev polynomial for all α when n is even, and for small α (specifically, $0 \le \alpha \le \sin(\pi/2n)$) when n is odd; for n odd and $\alpha > \sin(\pi/2n)$ the solution is complicated. See, e.g., Achiezer [1970, p. 209].

What we shall do is to discuss a simple problem, not directly relevant, but which illustrates the general method of solution and shows how to dispel some of the mysteries about [Z1 . The problem, which has been discussed by Hornecker [1958], Achiezer [1956, 1967], Bernstein [1926], Talbot [1962, 1964], is:

(B1) <u>Determine</u>

$$\min \max_{(a)\ 0\le x\le 1} |(1+x)^{-1} - (a_0x^n + \ldots + a_n)|.$$

For references to the literature on this problem, see Todd [1984b].

*$\tilde{I}$ denotes "curly theta".

A method of handling *all* the problems discussed is to guess the answer and then confirm it by appealing to the weighted rational equal ripple theorem, due essentially to Chebyshev but refined by de la Vallée Poussin and others. In the non-degenerate case we have:

Equal Ripple Theorem (ERT). Suppose $f(x)$ and $w(x)$ are continuous in $[a,b]$, and that $w(x) \neq 0$ in $[a,b]$. Then the extremal function for the problem

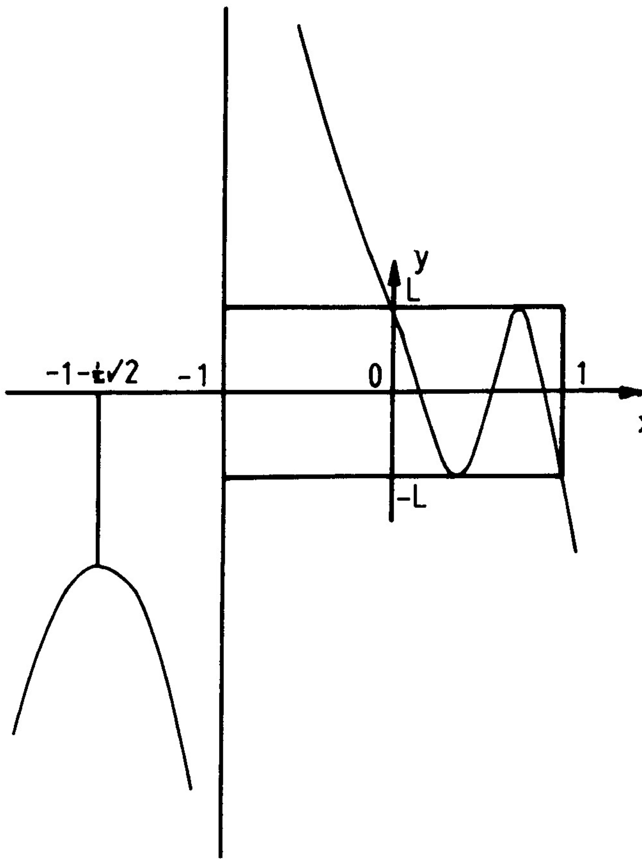
$$\min_{N,D} \max_{a \le x \le b} |E(x)|, \quad \text{where } E(x) = f(x) - w(x)\frac{N_n(x)}{D_d(x)}$$

(where $N_n(x) = b_0x^n + \ldots + b_n$, $D_d(x) = a_0x^d + \ldots + a_d, a_0 \neq 0$) is characterized by $E(x)$ assuming its maximum absolute value with positive and negative signs alternately, $n + d + 2$ times in $[a,b]$.

The original application of this is to (T1) (with $n + 1$ for n) when $f(x) = x^{n+1}$ and $w(x) \equiv 1$, $d = 0$, $a_0 = -1$. Another instance is when $f(x) = 1$, $w(x) = x^{-1/2}$ which occurs in §3C. Detailed knowledge of the trigonometric and elliptic functions ensures quick but unmotivated proofs. Thus H. Lebesgue [1920], reviewing de la Vallée Poussin's [1919] Borel Tract, writes "..par une sorte de divination qui rappelle bien son illustre compatriote Tchebycheff, M. Bernstein trouve les polynomes d'approximation de $(z-a)^{-1}$, ...".

We now illustrate, by discussing (B1), a second approach to our problems which involves the use of ERT at the start to get a differential equation for the extremal error which may then be solved and lead to the result required. In this approach we refer to tables of (elliptic) integrals instead of to the properties of elliptic functions.

What we apply used to be called "Curve Tracing": From the qualitative behavior specified by ERT we can get a differential equation for the solution to (B1). The even and odd cases look slightly different -- we shall deal only with the case $n = 2$. Using ERT we show that the graph of the error function $y(x)$ must be of the form:



$$y = [(1 + x)^{-1} - (a_0x^2 + a_1x + a_2)]$$

$$n + 2 = 4$$

$y \sim -a_0x^2$ at $\pm\infty$

$y \sim \pm\infty$ at $-1\pm$

y' three zeros

$y \pm L$ each three zeros

Figure 7

If the extremal values of $y(x)$ are $\pm L$ then $y^2 - L^2$ has simple zeros at $0,1$ and n double zeros in the interior of $[0,1]$ while y' has n simple zeros at these points and a single extraneous zero, α in $(-\infty, -1)$. It follows that

$$(1 + x)^2 x(1 - x) y'^2 = n^2(x - \alpha)^2(L^2 - y^2)$$

where α, L are yet to be found. Writing $y = L\eta$ and noting that $\eta(0) = 1$ we find

$$\int_1^\eta \frac{\pm dY}{\sqrt{1-Y^2}} = n \int_0^x [1 - \frac{1+\alpha}{1+X}] \frac{dX}{\sqrt{X(1-X)}}$$

where the ambiguous sign changes at each extrema, beginning with a negative sign.

The integrals involved here are elementary and we can solve explicitly for η. If we use the fact that $y(1) = -L$ we can determine

$$\alpha = \alpha_n = -1 - \sqrt{2}n^{-1}.$$

To determine L we use the fact that $(1+x)y(x) \sim 1$ as $x \sim -1$. This gives

$$L = L_n = \frac{1}{4} (3-2\sqrt{2})^n.$$

[The results for α_n, L_n for $n = 1, 2$ can be checked by elementary methods.]

The final result is that the best approximation is given by

$$\sqrt{2} \{[\tfrac{1}{2} - c\, T_1(2x - 1) + c\, T_2(2x - 1) + \dots +$$

$$+ (-1)^{n-1}c^{n-1}T_{n-1}(2x - 1)\} + (-1)^n(1 - c^2)^{-1}\, T_n\, (2x - 1)$$

where $c = 3 - 2\sqrt{2}$. This expression is remarkable because it is got by truncating the Fourier-Chebyshev expansion of $(1 + x)^{-1}$ and dividing the last term by $1 - c^2$. This was pointed out explicitly by Hornecker [1958] and Talbot [1962] and examined further by Rivlin [1962].

We now return to (Z1). From ERT it follows that the extremal $y(x) = x^n - n\sigma x^{n-1} + a_2 x^{n-2} + \ldots$ has $(n-2) + 2 = n$ alternating extrema. Curve tracing arguments show that either all $n - 1$ zeros of y', or all but one of these lie in $[-1,1]$, and this is the cause for the separation into the trigonometric and elliptic cases. We sketch the behavior of y, when $n = 2$ and $n = 3$, in the two cases:

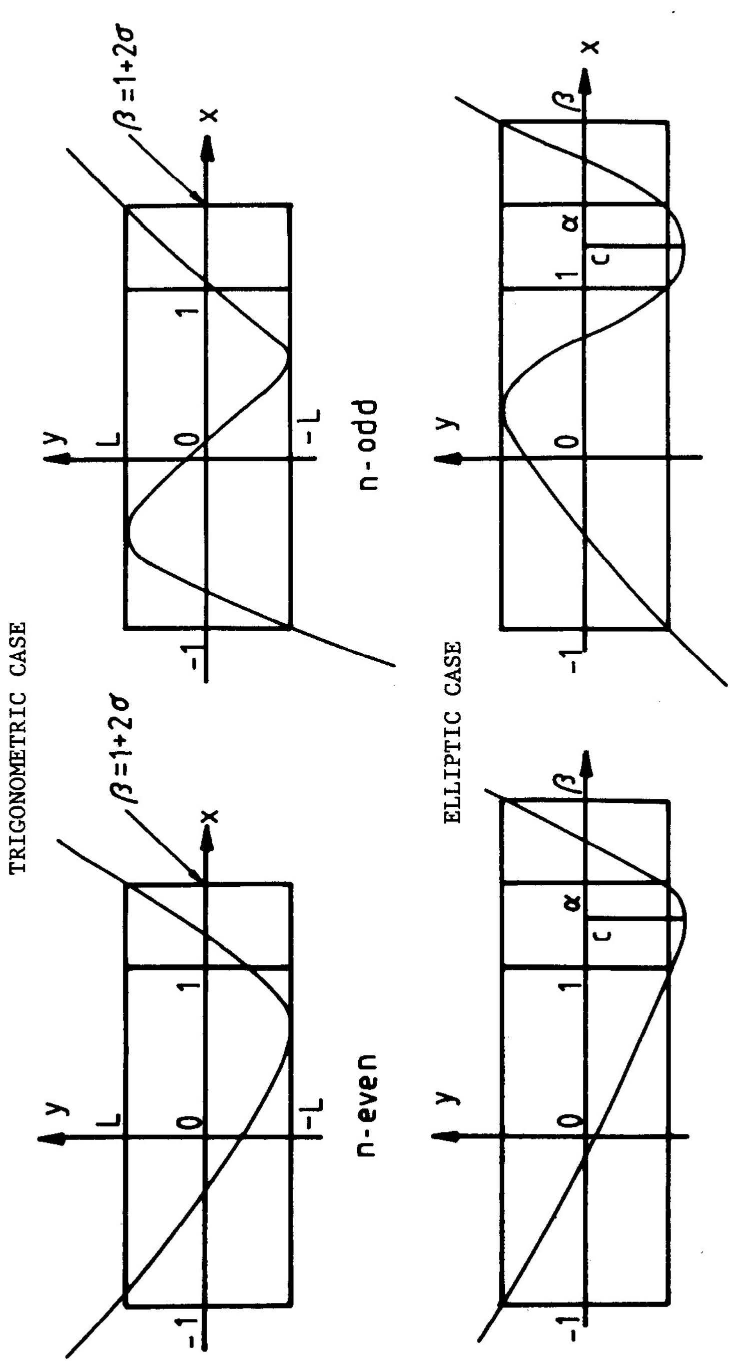

TRIGONOMETRIC CASE
ELLIPTIC CASE
n-odd
n-even
β=1+2σ
β=1+2σ
x
y
L
0
1
-1
-L
α
c
β

Figure 8

The differential equation in the trigonometric case is

$$\frac{\pm\, dy}{\sqrt{(L^2 - y^2)}} = \frac{n\,dx}{\sqrt{\{(1 + x)(\beta - x)\}}}$$

where the deviation L and the largest extrema, β, have to be determined.

The differential equation in the elliptic case is

$$\frac{\pm\, dy}{\sqrt{(y^2 - L^2)}} = \frac{n(c - x)dx}{\sqrt{\{(x^2 - 1)(\alpha - x)(\beta - x)\}}}$$

where the deviation L, the two largest extrema α, β and the extraneous turning point c have to be determined.

Alternative accounts of the solution of (Z1) are given by Erdös and Szegö [1942], by Achiezer [1953; 1967] and by Carlson and Todd [1983].

In the solution of (A1) the degenerate form of the ERT is required: if the polynomials N and/or D are truncated say

$$N_n = b_0 x^n + \ldots + b_{n-\mu} x^{n-\mu}, \quad D_d = a_0 x^d + \ldots + a_{d-\nu} x^\nu$$

then the number of extrema must be reduced by $\max(\mu, \nu)$. For a discussion of the general case of ERT see, e.g. Achiezer [1953, p. 55].

§5. THE GENERAL METHOD OF SOLUTION

The problems we have discussed led, by use of the Equal Ripple Theorem, to differential equations of the form

$$\frac{dx}{\sqrt{\{(1 - x^2)(1 - k^2x^2)\}}} = \frac{M\,dy}{\sqrt{\{(1 - y^2)(1 - \lambda^2 y^2)\}}} \tag{1}$$

where y was to be an algebraic, rational or polynomial function of x and where k was given and M, λ to be determined. In some cases k, or k and λ were zero. In some cases elliptic differentials of the third kind or hyper-elliptic differentials were involved instead of those of the first kind.

Thus our problems are those of the "transformation" of elliptic objects (differentials, integrals, functions or $\mathcal{I}$-functions). This theory has been with us since the very beginning with formulation and solutions by Abel, Jacobi, Gauss, Legendre, Riemann, etc.

The recent literature on transformation is not very extensive: Riemann [1899], Tricomi [1948], Achiezer [1970], Lang [1973], Rauch-Lebowitz [1973], Robert [1973], Houzel [1978].

We begin with four elementary examples: the first three deal with integrals.

(I) Fagnano (1682-1766) pointed out essentially that

$$\int_0^T \frac{dt}{\sqrt{(t(1-t^2))}} = \int_Z^1 \frac{dz}{\sqrt{(z(1-z)))}}$$

if $T = (1 - Z)/(1 + Z)$ which is established by the (linear) transformation

$$t = (1 - z)/(1 + z).$$

Hence if $T = \sqrt{2} - 1$ then $Z = \sqrt{2} - 1 = T$. This means we have succeeded in bisecting a quadrant of a lemniscate ($r^2 = \cos 2\theta$, in polar coordinates). We can represent the curve parametrically as

$$x = (t(1 + t)/2)^{1/2}, \quad y = (t(1 - t)/2)^{1/2}$$

and, differentiating, we find

$$\dot{s}^2 = \dot{x}^2 + \dot{y}^2 = \{4t(1 - t^2)\}^{-1}.$$

(II) Gauss (1777-1855) and Landen (1719-1790) essentially used a quadratic transformation to determine the arithmetic-geometric mean M of non-negative a_0, b_0 where $a_0 > b_0$. The existence of $M = \lim a_n = \lim b_n$, where a_n, b_n are defined by

$$a_{n+1} = \frac{1}{2}(a_n + b_n), \quad b_{n+1} = (a_n b_n)^{1/2},$$

is easily established by monotony. The transformation

$$y = \frac{1}{2}(x^2 - a_n b_n)/x$$

which runs from $-\infty$ to ∞ as x runs from 0 to ∞ gives, with a little algebra,

$$\int_{-\infty}^{\infty} \{(x^2 + a_n^2)(x^2 + b_n^2)\}^{-1/2}\, dx =$$

$$\int_{\infty}^{\infty} \{(y^2 + a_{n+1}^2)(y^2 + b_{n+1}^2)\}^{-1/2}\, dy.$$

Repetition of this gives

$$\int_{-\infty}^{\infty} \{(x^2 + a_0{}^2)(x^2 + b_0{}^2)\}^{-1/2}\, dx =$$

$$\int_{-\infty}^{\infty} \{(t^2 + M^2)(t^2 + M^2)\}^{-1/2}\, dt = \pi/M$$

so that

$$M = \frac{\pi}{\int_{-\infty}^{\infty} \{(x^2 + a_0{}^2)(x^2 + b_0{}^2)\}^{-1/2}\, dx} =$$

$$= \frac{\pi a_0}{2\, K\{(a_0{}^2 - b_0{}^2)/a_0{}^2)^{1/2}\}} .$$

This presentation is based on one of D. J. Newman [1982]; there are several others available, all depending on an invariant integral.

(III) Landen showed, by a geometric argument essentially equivalent to that in (II), that if

$$k_1 = (1 - k')/(1 + k')$$

then

$$K_1 = \frac{1}{2}(1 + k')\, K, \quad K_1' = (1 + k')\, K'.$$

(IV) The differential equation

$$n(1 - x^2)^{-1/2}\, dx = (1 - y^2)^{-1/2}\, dy, \quad y(1) = 1$$

is satisfied by the Chebyshev polynomial

$$y = T_n(x) = \cos(n \arccos x) .$$

It is not too surprising that the well-known approximation properties of this degenerate case of the transformation equation carry over in some measure to the general case.

The special case of (1), when $\lambda = k$, is called the multiplication problem. The solutions to this will either hold for all k and then $M = n$, a positive integer, or will hold only for special values of k, when the period ratio is a complex quadratic surd, as is the multiplier, M. The first type is called real multiplication, the second is called complex multiplication.

We pursue this question of transformation and multiplication further. The Weierstrassian case is trivial, since the periods of $\wp$ can be chosen arbitrarily (subject only to $(w_2/w_1) > 0$). Thus if $\wp$ has periods $2w_1$, $2w_2$ then

$$\wp(z) + (e_1 - e_2)(e_1 - e_3)(\wp(z) - e_1)^{-1}$$

has periods w_1, $2w_2$. [Cf. W. & W, 456, 444.] Also [W & W, 441]

$$\wp(2u) = \frac{\wp^4 + \frac{1}{2} g_2\wp^2 + 2g_3\wp + \frac{1}{16} g_2^3}{4\wp^3 - g_2\wp - g_3}$$

where the arguments u of the $\wp$'s on the right have been omitted.

There is a very different state of affairs in the Jacobian case, for the quarter-periods K, K' are not independent, both being uniquely determined by k. We cannot construct a Jacobian function with say, quarter periods, K_1, K_1' with $K_1' = 2K'$, $K_1 = K$; the best we can do is to introduce a "multiplier" for the argument. We discuss a numerical example.

As k increases from 0 to 1, K increases from $\frac{1}{2}\pi$ to ∞ and K' decreases from ∞ to $\frac{1}{2}\pi$. Consequently, K'/K decreases from ∞ to 0. Here are two sets of numerical values:

$k = 0.8$, $k' = 0.6$, $K = 1.9953$, $K' = 1.7508$, $K'/K = 0.8774$.

$k = 0.25$, $k' = 0.9682$, $K = 1.5962$, $K' = 2.8012$, $K'/K = 1.749$.

These are special cases of the (complete) Landen transformations (W&W, §22.42) mentioned in Example III above.

If we begin with $\mathrm{sn}(u, k)$, with $k = 0.8$ and $K'/K = 0.8774$ it is clear from the graph of K'/K against k that there will be a unique λ, actually $\lambda = 0.25$, such that $\Lambda'/\Lambda = 2\ K'/K$ and that Λ/K has a specific value, actually 0.8, the multiplier, M. The elliptic function

$\mathrm{sn}(u/M, \lambda)$, actually $\mathrm{sn}(1.25\ u, 0.25)$

will have quarter periods K, $2K'$.

In general the multiplier will depend on n, the order of transformation and on the modulus, k.

We now outline the solution of the general transformation problem. There are some advantages in using the Riemann Normal Form. We shall, however, for simplicity use the Weierstrass form. But, for applications, we will return to the traditional Jacobi form, despite its complications.

Take the case

$$u = \int_0^X \frac{dX}{\sqrt{(4X^2 - g_2X - g_3)}}, \quad v = \int_0^Y \frac{dY}{\sqrt{(4Y^2 - \gamma_2 Y - \gamma_3)}} \qquad (2)$$

with

$u = Mv$.

Inverting (2) we get

$x = (u;\ 2w_1,\ 2w_2)$, $y = (v;\ 2\Omega_1,\ 2\Omega_2)$

where x has periods $2w_1$, $2w_2$ and y has periods $2\Omega_1$, $2\Omega_2$. Because of the hypothesis that x, y are related by a polynomial equation we can conclude that the periods are not independent and that the elliptic functions x, y have a common period parallelogram and, say,

$$\begin{aligned} r\Omega_1 &= aw_1 + bw_2 \\ s\Omega_2 &= cw_1 + dw_2 \end{aligned} \qquad (3)$$

where r, s, a, b, c, d are integers. An algebraic relation between x, y can be obtained by taking a polynomial of sufficiently high degrees in x, y, balancing the principal parts by choice of the coefficients, and then appealing to Liouville's Theorem. (Cf. e.g. Copson, [1935].) We can show that if the relation is $A(x, y) = 0$ then to a value of x corresponds r values of y and to a value of y corresponds $n = ad - bc$ values of x.

We shall now show that the algebraic relation between x, y can be replaced by two rational ones. In fact denote by z the γ function with half-periods $\pi_1 = r\,\Omega_1$, $\pi_2 = s\,\Omega_2$ and apply the last remark in the immediately preceding paragraph to the pair x, z and to the pair y, z. We conclude that x, z are connected by an equation

$$B(x,z) = 0$$

which is of degree $1 \times 1 = 1$ in x and degree rs in z and that y, z are connected by an equation

$$C(x, z) = 0$$

of degree 1 in y and degree n in z. So x and y are expressible rationally in terms of the new variable z.

Our problem now is to find in closed form the transformation between the two elliptic objects (integrals, functions (Weierstrass or Jacobian) or theta functions) with which we are concerned.

We have seen that it is enough to discuss the rational case when the half-periods are related by

$$\left.\begin{aligned} \Omega_1 &= aw_1 + bw_2 \\ \Omega_2 &= cw_1 + dw_2 \end{aligned}\right\}$$

with a, b, c, d integers (without common factor) and $ad - bc = n > 0$.

To do this we consider the factorization of the matrices

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} .$$

First of all, any such matrix M can be represented as a product of the form

$$U^{\cdot}S^{\cdot}\ U^{\cdot}S^{\cdot}\ \ldots\ N\ \ldots\ S^{\cdot}U^{\cdot}S^{\cdot}U^{\cdot}$$

where

$$N = \begin{bmatrix} n & 0 \\ 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

and where the powers to which U, S are raised are not indicated, except by dots. (This result is due to C. Cellitti [1914].) For instance,

$$\begin{bmatrix} 3 & 4 \\ 2 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^3 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^7 \begin{bmatrix} 52 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-7} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} .$$

We can now factorize N into diagonal matrices using the decomposition of n into prime factors. Continuing our example

$$\begin{bmatrix} 52 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^2 \begin{bmatrix} 13 & 0 \\ 0 & 1 \end{bmatrix} .$$

Thus M can be represented as a product of 2×2 matrices of determinant 1, 2 and odd primes. The transformation is similarly decomposed into transformations of order 1, order 2, and order an odd prime. The last two are called principal transformations of the first kind -- those of the second kind correspond to matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix}$$

which can however, be represented in terms of matrices of the first kind and the matrix of order 1

$$T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

since

$$T^{-1}\begin{bmatrix}1 & 0\\ 0 & n\end{bmatrix} T = \begin{bmatrix}n & 0\\ 0 & 1\end{bmatrix}.$$

Notice that a principal transformation means that one or other period is multiplied or divided by the order n and that a principal transformation of the first kind followed by one of the second kind, both of the same order produce a multiplication (division) of the periods by the order.

There are various ways to derive the relevant transformation formulas, e.g. the usual Liouville arguments or the use of the elementary multiple angle formulas and the definitions of the elliptic functions in terms of $\tilde{\mathcal{I}}$-functions. We outline the latter approach.

<u>Lemma</u>. $x^{2n} - 2x^n \cos n\theta + 1 = \prod_{s=1}^{n} \{x^2 - 2x\cos(\theta+(s\pi/n))+1\}$.

<u>Proof</u>. The left hand side is $(x^n - \exp(in\theta))(x^n - \exp(-in\theta))$. Use Demoivre's Theorem for each factor and then combine conjugate factors.

If in the classical formula ($\phi(q)$ standing for $\Pi(1-q^{2r})$)

$$\tilde{\mathcal{I}}_4(z,q) = \phi(q) \prod_{r=1}^{\infty} \{1 - 2q^{2r-1}\cos 2z + q^{4r-2}\}$$

we replace z by nz, and q by q^n and use the lemma in each factor on the right, change the order of multiplication, and recombine factors we find

$$\tilde{\mathcal{I}}_4(nz,q^n) = \{\phi(q^n)/[\phi(q)]^n\} \prod_{s=0}^{n-1} \tilde{\mathcal{I}}_4(z+(s\pi/n),q).$$

Similar results hold for the other $\tilde{\mathcal{I}}$-functions.

We now use the representations of the Jacobian elliptic functions as quotients of theta functions, such as

$$\mathrm{sn}(u,k) = \frac{\tilde{\mathcal{I}}_3}{\tilde{\mathcal{I}}_2} \cdot \frac{\mathcal{I}_1(u\tilde{\mathcal{I}}_3^{-2})}{\tilde{\mathcal{I}}_4(u\tilde{\mathcal{I}}_3^{2})}$$

to get transformation formulas for the Jacobian functions, in particular the following result.

Theorem. *If* n *is odd*

$$\mathrm{sn}(uM^{-1},\lambda^2) = M^{-1}\mathrm{sn}(u,k^2)\ \Pi\ \frac{1-c_{2s}^{-1}\mathrm{sn}^2(u,k)}{1-k^2c_{2s}\mathrm{sn}^2(u,k)} \tag{9}$$

where $c_r = \mathrm{sn}^2(rKn^{-1},k)$, *and*

$$\lambda = k^n\ \Pi\ c_{2r-1}^2,\quad M = \Pi(c_{2r-1}/c_{2r})$$

(*all products are over* s, *from* $s = 1$ *to* $s = \frac{1}{2}(n-1)$).

The transformation (9) is a First (Principal) Transformation: The periods Λ, Λ' being connected with K, K' by the relations

$$\Lambda = K/nM,\quad \Lambda' = K'/M.$$

In the Second (Principal) Transformation the periods are connected by the relations

$$\Lambda = K/M,\quad \Lambda' = K'/nM.$$

For details of these see e.g. Cayley [1895] and Achiezer [1970, p. 284].

We note here that in Greenhill [1892] there is a proof of the basic formulas by means of an electromagnetic analogy (Kelvin's method of images).

This formula (9) is essentially that used in the discussion of the Cauer problem (see, e.g. Oberhettinger and Magnus [1949]). To solve the ADI problem we have to use the n even analog of (9). The analog of (9) for the dn function gives the solution to the Ninomiya problem given above; we continue this discussion as announced.

Theorem. *If* n *is an integer then*

$$\mathrm{dn}(u\ M^{-1},\ \lambda) = \mathrm{dn}(u,k)\ \Pi\ \frac{c_{2m-1} + s_{2m-1}\ \mathrm{dn}^2(u,k)}{c_{2m} + s_{2m}\ \mathrm{dn}^2(u,k)} \tag{10}$$

where

$$\lambda = k^n \, \pi \, s^2_{2m-1}, \quad M = \pi(s_{2m-1}/s_{2m}),$$

$$s_j = \mathrm{sn}^2(jK/n,\, k), \quad c_j = \mathrm{cn}^2(jk/n,\, k),$$

and where all products run from $m = 1$ to $m = [n/2]$.

The optimal starting value, for $\sqrt{x}$, when x is restricted to $[a, 1]$, is given by

$$y = \sqrt{x/\lambda'}\ \mathrm{dn}(uM^{-1}, \lambda)$$

when

$$x = a\ \mathrm{nd}^2(u,k).$$

[As u runs from 0 to K, dn runs from 1 to k' and x from a to a/k'^2 so that we take $k' = \sqrt{a}$.]

If we specialize this to $n = 4$, $a = 1/4$ we find

$$y = \frac{(c_1 x + \frac{1}{4} s_1)(c_3 x + \frac{1}{4} s_3)}{\frac{1}{2}\sqrt{\lambda'}\ (c_2 x + \frac{1}{4} s_2)}$$

since $c_4 = 0$, $s_4 = 1$. When $k' = \sqrt{a} = \frac{1}{2}$ since

$$s_2 = (1 + k')^{-1}, \quad c_2 = k'(1 + k')^{-1}$$

we have

$$y = \frac{(2/c_2\sqrt{\lambda'})\ [c_1 c_3 x^2 + \frac{1}{4} x(s_1 c_3 + s_3 c_1) + (s_1 s_3/16)]}{x + \frac{1}{4}}.$$

From Ninomiya [1970] or Carlson and Todd [1983] we can compute the coefficients in the numerator. These are all surds and, e.g., we find the coefficient of x^2 to be

$$\alpha = 2(\sqrt{2} - 1)(24\sqrt{2})^{-1/4} = 0.34322\ \ 0129;$$

that of x is

$$\alpha(3 + \sqrt{2})/\sqrt{2} = 1.07129\ 9971$$

and the constant term is

$$\frac{1}{4}\alpha = 0.08580\ 5032.$$

These values agree with those in the continued fraction representation given by Ninomiya [1970, p. 403].

§6. A MODERN TREATMENT OF TRANSFORMATION

This is described in the books of Houzel [1978], Lang [1973] and Robert [1973]. It is somewhat sophisticated and the essential identity of elliptic functions, the tori which are the corresponding Riemann surfaces, elliptic curves and period lattices is assumed.

A singular cubic curve is rational, i.e. its points can be expressed rationally in terms of a parameter, e.g. the slope of a line through the singularity (such a line meeting the curve in one other point). To parameterize a nonsingular cubic, we need elliptic functions. For the curve

$$y^2 = 4x^3 - g_2x - g_3$$

or, in homogeneous form,

$$y^2z = 4x^3 - g_2xz^2 - g_3z^3$$

we can take

$$x = \wp(u),\ y = \wp'(u) \tag{1}$$

where the $\wp$ function has invariants g_2, g_3. We can define an abelian group on such a curve by the following geometrical construction, when a point 0 on the curve is taken arbitrarily as the zero element: to find the sum C of points A, B on the curve we denote by D the residual intersection of the line AB and then the sum "$A + B$" is the residual intersection of OD with the curve. It is easy to see that if 0 is taken as a point at infinity $(0,1,0)$ then if A has parameter a and B parameter b in representation (1) above then the parameter of $C =$ "$A + B$" is $a + b$.

It is natural to study mappings between two elliptic curves (or their lattices) which preserve the group operation -- these are called isogenies. It turns out that these are essentially the transformations which we have been studying.

§7. REMARKS

1. There have been discussions of the Zolotarev problems when the L_1 norm is used in place of the L_∞, when trigonometric polynomials, and when entire functions are used in place of polynomials. See e.g. Gončar [1969], Ryžakov [1965, 1969], Meiman [1960, 1962], Galeev [1975], Feherstorfer [1979].

2. The elegant solutions to the problems can be used to indicate the efficiency of algorithms for optimal parameters in cases where there are no theoretical results available. Of course, from general principles, near optimal parameters will give very near optimal results, in a smooth environment.

3. There has been a certain amount of activity in other aspects of rational approximation, e.g. by D. J. Newman [1978], A. R. Reddy [1977, 1978], E. B. Saff and R. S. Varga [1980]. Particulary relevant is recent work by Lorentz, Saff, Varga and others on approximation by incomplete polynomials.

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* This study is a sketch for part of an extensive survey article "Applications of elliptic functions and elliptic integrals" which will appear elsewhere. A preliminary version was presented at a special session on "History of Contemporary Mathematics" at the Annual Meeting of the American Mathematical Society, 7 January 1981.

REFERENCES

1. Achiezer, N. I., "Vorlesungen über Approximations-theorie", Akademic Verlag, Berlin, 1953, translated as, "On the Theory of Approximation", Ungar, New York, 1956. (Second German edition, Berlin, 1967.) "Elements of the Theory of Elliptic Functions", 2nd edition (in Russian), 1970. For complete bibliographies of Achiezer see the articles on him in Uspehi Mat. Nauk ≡ Russian Math. Surveys, 16, 6, (102), 1961; p. 14; 26, 6, (162), 1971, pp. 257-261; 36; 4, (220), 1981, pp. 183-4.

2. Amer, R. A., Schwarz, H. R., "Contributions to the approximation problem of electrical filters", Mitt. Inst. ang. Math., ETH, Zürich, #9, 1964.

3. Bateman Project, "Higher Transcendental Functions, II", McGraw Hill, New York, 1953.

4. Bernstein, S., "Lecons sur les propriétés exctrémales et la meilleure approximation des fonctions analytiques d'une variable réelle", Gauthier-Villars, Paris, 1926. Chel., 1970.

5. Blum, E. K., "Numerical analysis and computation, theory and practice", Addison-Wesley, 1972.

6. Borwein, J. M., and Borwein, P. B., "Elliptic integrals and approximations to π", to appear; Cubic and Higher Order Algorithms for π, Canada Math. Bull. to appear; Explicit algebraic nth order approximations to π, this volume; The arithmetic geometric mean and fast computation of elementary functions, SIAM Rev., to appear.

7. Brent, R. P., "Fast multiple precision evaluation of elementary functions", J. ACM 23, 1976, pp. 212-251.

8. Carlson, B. C., and Todd, J., "The degenerating behavior of elliptic functions", SIAM J. Numerical Anal. 20, 1983, pp. 1120-1129; "Zolotarev's First Problem - the best approximation by polynomials of degree $\le n - 2$ to $x^n - n E x^{n-1}$ in $[0, 1]\sigma]$, Aq. Math., 1984, 26 (1983, pp. 1-33.

9. Cauer, W., "Synthesis of linear communications networks", McGraw Hill, New York, 1958, German edition 1934 "Bemerkung über eine Extremalaufgabe von E. Zolotareff" ZAMM 20, 1940, 358.

10. Cayley, A., "Elliptic functions", Deighton Bell, Cambridge 1895.

11. Cellitti, C., "Nuova representazione della sostituzione lineare binaria primitua, Atti Reale Acc. deiLincei. Ser. V, 23, 1914, pp. 208-213.

12. Chebyshev, P. L., "Oeuvres, I, II", St. Petersburg, 1899, Chelsea, New York.

13. "Sur les fractions algébriques...", Bull. Soc. Math. France, 12, 1884, pp. 167-168 (≡II, 725).

14. Cody, W. J., "Double-precision square root for the CDC-300", Comm. ACM 7, 1964, pp. 715-718.

15. Copson, E. T., "Functions of a complex variable", Oxford, 1935.

16. Curtis, A. R., "Tables of Jacobian elliptic functions whose arguments are rational fractions of the quarter period", NPL Math. Tables, v. 7, H. M. Stationery Office, London, 1964.

17. De Boor, C., and Rice, J. R., "Chebyshev approximation by a $\pi[(x-x_i)/(x-s_i)]$ and application to ADI iteration", J. SIAM, 11, 1963, pp. 159-169.

18. de la Vallée Poussin, C. J., "Lecons sur l'approximation des fonctions d'une variable réelle",Gauthier-Villars, Paris, 1919, Chelsea, 1970.

19. Dixon, A. C., "The elementary properties of elliptic functions", Macmillan, London, 1894.

20. Dumas, S., "Sur le développement des functions elliptiques en fractions continues", Thesis, Zürich, 1908.

21. Darlington, S. D., "Synthesis of resistance 4-poles which produce prescribed insertion loss characteristics including special application to filter design", J. Math. Phys., 18, 1939, pp. 257-353.

22. Enneper, A., rev. F. Müller, "Elliptische Functionen, Theorie und Geschichte", Halle, 1890.

23. Erdös, P., and Szegö, G., "On a problem of I. Schur", Ann. of Math. {2}43, 1942, pp. 451-470.

24. Farhutcinova, R. F., "Approximate analytic formulas for coefficients of Zolotarev polynomials and the process of economization via Zolotarev polynomials", p. 126-135 in Uniform approximations and the moment problem, ed. M. Ja. Zinger, Dal´nevostocn. Naucn. Centr. Akad. Nauk, SSSR, Vladivostok., 1977.

25. Fricke, R., "Lehrbuch der Algebra", 1924.

26. Gaier, D., Todd, J., "On the rate of convergence of optimal ADI processes", Numer. Math. 9, 1967, pp. 452-459.

27. Galeev, E. M., "Zolotarev Problem in the Metric of $L_1(-1,1)$", Mat. Zametki 17, 1975, pp. 13-20 = Math. Notes, 1975, pp. 9-13.

28. Gastinel, N., "Sur le meilleur choix des paramètres de sur-relaxation", Chiffres 5, 1962, pp. 109-126.

29. Gautschi, W., "Computational methods in special functions", pp. 1-98 in Theory and Application of Special Functions, Academic Press, 1975.

30. Glowatski, E., "Sechstellige Tafel der Cauer-Parameter", Abh. Bayer. Akad. Wiss., Math. Nat. Klasse, 67, 1955.

31. Goncar, A. A., "The problems of E. I. Zolotarev which are connected with rational functions", Math. Sb. (N.S.), 78, 120, 1969, pp. 640-654 translated in USSR - Sb. 7, 1969, pp. 623-625.

32. Greenhill, G., "The Applications of Elliptic Functions", Macmillan, London, 1892.

33. Hornecker, G., "Évaluation approcheé de la meilleure approximation polynomiale d'ordre n de f(x) sur un segment fini [a,b]", Chiffres 1, 1958, pp. 157-169.

34. Houzel, C., "Fonctions elliptiques et intégrales abéliennes", in Abrégé d'histoire des mathématiques 1700-1900, J. Dieudonné, directeur de la publication, 2 vols., Hermann, Paris, 1978.

35. King, L. V., "On the direct numerical calculation of elliptic functions and integrals", Cambridge, 1924.

36. Kuznetsov, P. I., "D. F. Egorov (on the centenary of his birth)", Usp. mat. nauk. = Russian Math. Surveys 26, 1971, #5, pp. 124-164.

37. Lang, S., "Elliptic Functions", Addison-Wesley, Reading, Mass., 1973.

38. Lebedev, V. I., "On a Zolotarev problem in the method of alternating directions", Z. Vycisl. Mat. i Mat. Fiz., 17, 1977, pp. 349-366, translated in USSR Comp. Math. and Math. Phys., 17, 1977, pp. 58-76, 1978.

39. Lebesgue, H., "Review of de la Vallée Poussin [1919]", Bull. Sc. Math. {2}44, 1920, pp. 137-153 ≡ Oeuvres Scientifiques 5, 1973, pp. 322-338.

40. Magnus, W., Oberhettinger, F., and Soni, R. P., "Formulas and theorems for the special functions of mathematical physics", 3, Springer, Heidelberg, 1966.

41. Markushevich, A. I., "The remarkable sine functions", Elsevier, New York, 1966.

42. Meiman, N. N., "Solution of the fundamental problems of the theory of polynomials and entire functions which deviate least from zero", (in Russian), Trudy Moskov. Math. Obsc. 9, 1960, pp. 507-535, MR24A#2031.

43. Meiman, N. N., "On the theory of polynomials deviating least from zero", 1960, translated in Soviet Math. Dokl. 1, pp. 41-44.

44. Meiman, N. N., "Polynomials deviating least from zero with an arbitrary number of given coefficients", 1960, translated in Soviet. Math. Dokl. 1, pp. 72-75.

45. Meiman, N. N., "The zeros of a class of multiple valued-functions", translated in AMS Trans. 2, 19, 1962, pp. 167-171.

46. Meiman, N. N., "The theory of functions of class HB and B^x", translated in AMS Trans. (2), 19, 1962, pp. 173-178.

47. Meinardus, G., "Approximation von Funktionen und ihre numerische Behandlung", Springer, 1964, translated by L. L. Schumaker as "Approximation of Functions, Theory and Numerical Methods", Springer, 1967.

48. Meinardus, G., and Taylor, G. D., "Optimal partitioning of Newton's method for calculating roots", Math. Comp. 35, 1980, pp. 1231-1250.

49. Meinguet, J., and Belevitch, V., "On the realizability of ladder filters", IRE Trans. Prof. Groups on Circuit Theory, CT 5, 1958, pp. 253-255.

50. Melzak, Z. A., "Mathematical ideas, modeling and applications", Vol. II of Companion to concrete mathematics, Wiley, New York, 1976.

51. Milne-Thomson, L. M., "Jacobian elliptic function tables", Dover, New York, 1950.

52. Moursund, D. G., "Optimal starting values for Newton-Raphson calculations of $\sqrt{x}$", Comm. ACM 10, 1967, pp. 430-432.

53. Moursund, D. G., and Taylor, G. D., "Optimal starting values for the Newton-Raphson calculation of inverses of certain functions", SIAM J. Numer. Anal. 5, 1968, pp. 138-150.

54. NBS Handbook of Mathematical Functions, ed. M. Abramowitz - I. A. Stegun, U. S. Government Printing Office, Washington, D.C., 1964.

55. Neville, E. H., "Jacobian elliptic functions", Oxford, 1946.

56. Newman, D. J., "Approximation with rational functions", CBMS, Regional conference series in mathematics #41, 1978, American Math. Soc., Providence, R.I.

57. Newman, D. J., "Rational approximation versus fast computer methods", in Lectures on approximation and value distribution, Univ. de Montréal, 1982.

58. Ninomiya, I., "Generalized rational Chebyshev approximation", Math. Comp. 24, 1970, pp. 159-169. "Best rational starting approximations and improved Newton iteration for the square roots", Math. Comp. 24, 1970, pp. 391-404.

59. Oberhettinger, F., and Magnus, W., "Anwendung der elliptischen Funktionen in Physik und Technik", Springer-Verlag, Heidelberg, 1949.

60. Paszkowski, S., "The theory of uniform approximation", I, Rozprawy Matem., 25, Warsaw, 1962.

61. Piloty, H., "Zolotareffsche rationale Funktionen", ZAMM 34, 1954, pp. 175-189.

62. Ozigova, E. P.,"Zolotarev, E. I., 1847-1878",Nauka, Moscow, 1966 (in Russian).

63. Peaceman, D. W., and Rachford, H. H., Jr., "The numerical solution of parabolic and elliptic differential equations", J. SIAM 3, 1955, pp. 28-41.

64. Feherstorfer, F., "On the representation of extremal functions in the L^1-norm", J. Approx. Theory, 27, 1979, pp. 61-75.

65. Pokrovskii, V. L., "On a class of polynomials with extremal properties", Mat. Sb. (N.S.) 48, 90, pp. 257-276, translated in AMS Trans. (2), 19, 1962, pp. 199-219.

66. Rauch, H. E., and Leibowitz, A., "Elliptic functions, theta-functions and Riemann surfaces", Baltimore, 1973.

67. Rivlin, T. J., "Polynomials of best approximation to certain rational functions", Numer. Math. 4, 1962, pp. 345-369.

68. Robert, Alain, "Elliptic curves", Springer Lecture Notes in Mathematics, #326, 1973.

69. Reddy, A. R., "Rational approximation to $x^{n+1} - \sigma x^n$", Quart. J. Math. (Oxford), 2, 28, 1977, pp. 123-127.

70. Reddy, A. R., "A note on a result of Zolotarev and Bernstein", Manuscripta Math. 10, 1977, pp. 95-97.

71. Reddy, A. R., "On certain problems of Chebyshev, Zolotarev, Bernstein and Akhieser", Invent. Math. 45, 1978, pp. 83-110.

72. Rice, J. R., "The approximation of functions", 2 vols., Addison-Wesley, 1964, 1969.

73. Riemann, B., "Elliptische Functionen", ed. H. Stahl, Teubner, Leipzig, 1899.

74. Rivlin, T. J., "An introduction to the approximation of functions", 1969, 1981. Dover, N.Y.

75. Ryzakov, I. Ju., "An analog of a problem of E. I. Zolotarev", DAN USSR 160, 1965, 552-559 ≡ Soviet Math. Dokl. 6, 1965, pp. 157-159.

76. Ryzakov, I. Ju., "The trigonometrical analogue of a problem of E. I. Zolotarev", Izv. Vyss. Ucebn. Zaved. Mat. 8, 87, 1969, pp. 75-88.

77. Saff, E. B., and Varga, R. S., "Remarks on some conjectures of G. G. Lorentz", J. Approx. Theory 30, 1980, pp. 29-36.

78. Salamin, E., "Computation of π using arithmetic-geometric mean", Math. Comp. 30, 1976, pp. 565-570.

79. Stiefel, E. L., "Le problème d'approximation dans la théorie des filtres électriques", pp. 81-87, Belgian Colloquium, 1961.

80. Talbot, A., "On a class of Tchebysheffian approximation problems solvable algebraically", Proc. Cambridge Phil. Soc. 58, 1962, pp. 266-267.

81. Talbot, A., "The Tchebysheffian approximation of one rational function by another", Proc. Cambridge Phil. Soc. 60, 1964, pp. 877-890.

82. Talbot, A., Inaugural Lecture, "Approximation Theory or a miss is better than a mile", 18 November 1970, University of Lancaster Library, 1971.

83. Todd, J., "Introduction to the constructive theory of functions", Birkhäuser, Academic Press, 1963.

84. Todd, J., "Optimal ADI-parameters", in Functional Analysis, Approximationstheorie, Numerische Mathematik, ISNM, 7, Birkhäuser, Basel, 1967.

85. Todd, J., "Inequalities of Chebyshev, Zolotarev, Cauer and W. B. Jordan", pp. 321-328 in Inequalities, II., ed. O. Shisha, Academic Press, 1967.

86. Todd, J., "Optimal parameters in two programs", Bull. Inst. Math. Appl. 6, 1970, pp. 31-35.

87. Todd, J., "The lemniscate constants", Comm. ACM 18, 1975, pp. 16-19.

88. Todd, J., "The many values of mixed means", I., pp. 5-22, in General Inequalities, ed. E. F. Beckenbach, ISNM 41, Birkhäuser, Basel, 1978.

89. Todd, J., "Some applications of elliptic functions and integrals", Coll. Math. Soc. Janos Bolyai, vol. 22, Numerical Methods, ed. P. Rózsa, Keszthely (Hungary), 1977, pp. 591-618, North Holland, New York, and Amsterdam, 1980.

90. Todd, J., "The best polynomial approximation to $(1+x)^{-1}$ in [0,1]", to appear.

91. Tricomi, F. G., and Kraft, M., "Elliptische Funktionen", Leipzig, 1948.

92. Varga, R. S., "Matrix iterative analysis", Prentice-Hall, Englewood Cliffs, 1962.

93. Voronowskaja, E. V., "The functional method and its applications", Trans. Russian Math. Monographs, 28, 1970, American Math. Soc., Providence, R.I.

94. Wachspress, E. L., "Extended application of ADI iteration model problem theory", J. SIAM 11, 1963, pp. 994-1016.

95. Wachspress, E. L., "Optimum ADI iteration parameters for a model problem", J. SIAM 10, 1962, pp. 339-350.

96. Wachspress, E. L., "Iterative solution of elliptic systems", Prentice-Hall, Englewood Cliffs, 1966.

97. Whittaker, E. T., and Watson, G. N., "Modern analysis", Cambridge Univ. Press, 1927.

98. Zolotarev, E. I., "Complete collected works", I, 1931, II, 1932, Leningrad,

99. Zolotarev, E. I., "Sur l'application des functions elliptiques aux questions de maxima et minima", Bull. Acad. Sc. St. Petersburg 3, 24, 1878, 305-310, Melanges, 5, pp. 419-426 (≡I, pp. 369-376).

100. Zolotarev, E. I., "Anwendungen den elliptischen Functionen auf Probleme über Functionen die von Null am wenigsten oder am meisten abweichen"(in Russian), Abh. St. Petersburg 30, 1877, 5. (≡II, 1-59). See also remarks by N. G. Chebotarev and N. I. Achieser, II, pp. 357-361, Ob odnom voprose o naimehlshix velichnax, Diss. 1868, (≡II, 130-166).